ee: P34

1.6 元素訊號(Elementary signals)

- 指數訊號(exponential signals)
- 弦波訊號(sinusoidal signals)
- 弦波訊號與複數指數訊號
- 指數衰退之弦波訊號
- 步階(Step)函數(訊號)
- 脈衝(impulse)函數(訊號)
  - 微分
- RAMP function

指數訊號(exponential signals)

ee: P34

- (a) 自然界中常見之衰退或成長函數, 如: 人口自然成長函數, 放射物質之衰退
- (b)基本函數型態

$$x(t) = Be^{at},$$

$$B: t = 0 \text{之初值}$$

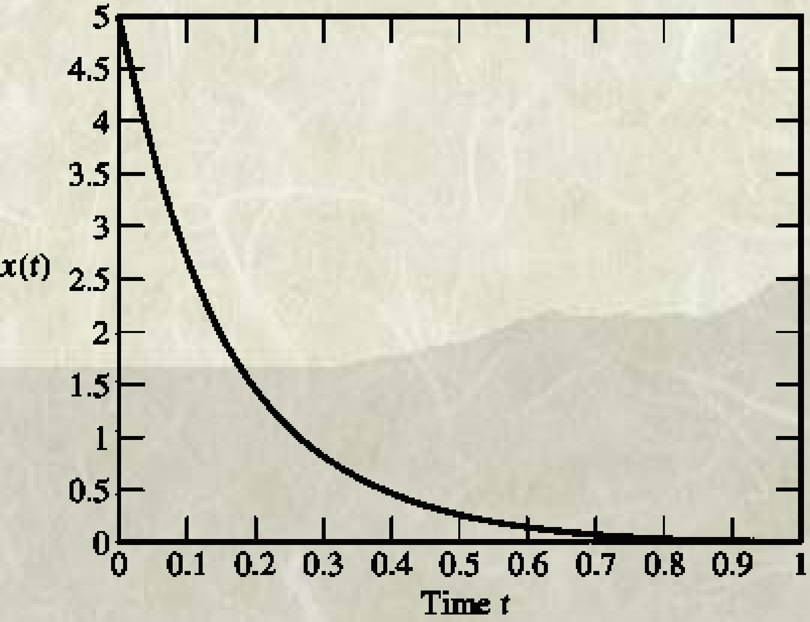
$$a: \text{指數蛻變常數}$$

$$a > 0, \text{成長}$$

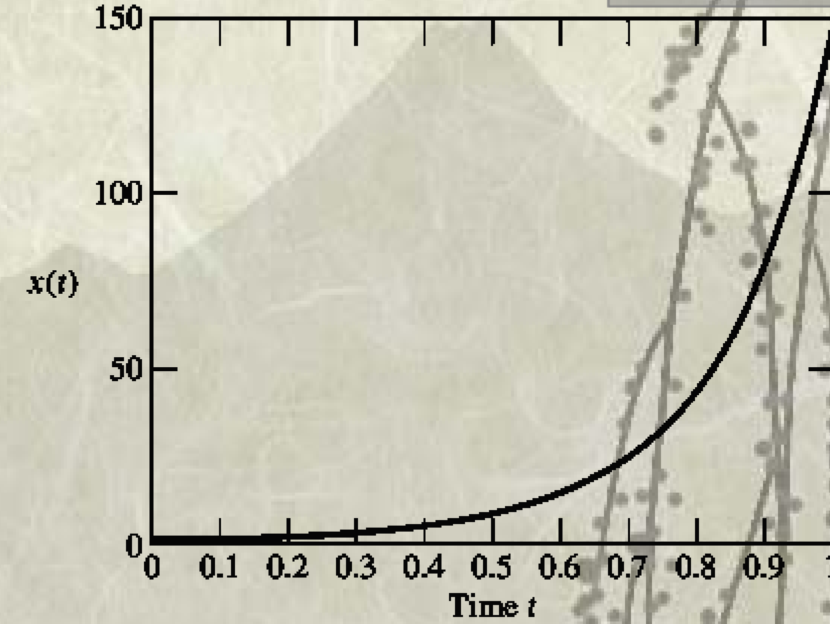
$$a < 0, \text{衰退}$$

圖例: 指數訊號(p34, F1.28)

ee: P34

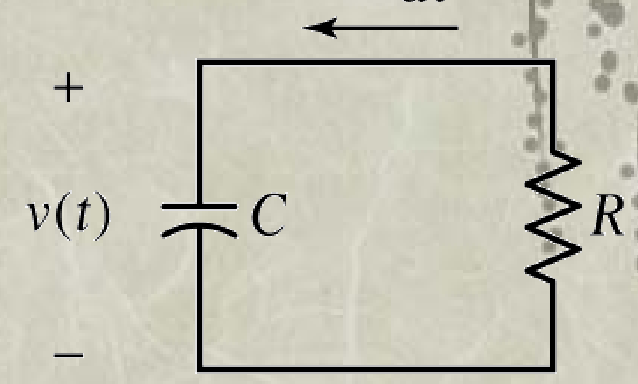


(a)



(b)

$$i(t) = C\frac{d}{dt}v(t)$$



- (a) $x(t) = Be^{-t/\tau}, \tau > 0$
- (b) $x(t) = Be^{t/\tau}, \tau > 0$

計算: 一階RC 電路

ee: P35

$$Ri(t) + v(t) = 0$$

$$RC\frac{d}{dt}v(t) + v(t) = 0$$

(一階常微分方程式)

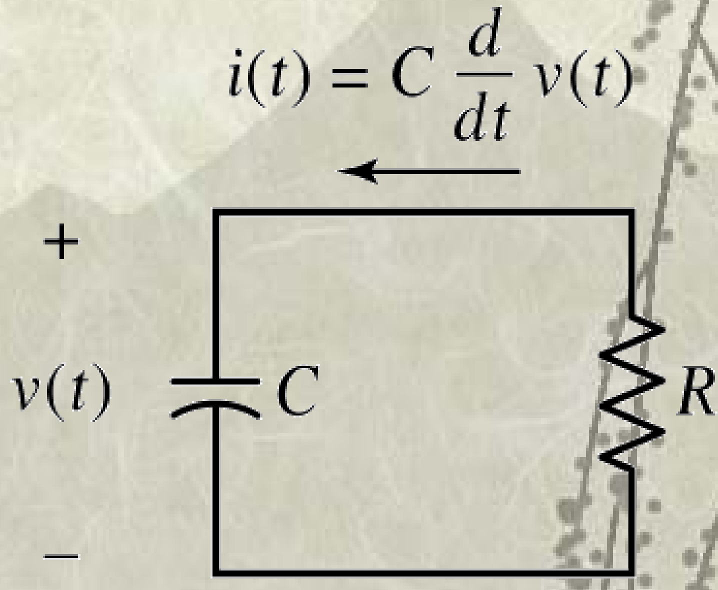
假設$v(t) = Be^{at}$

$$RC\frac{d}{dt}v(t) + v(t) = 0$$

$$RCaBe^{at} + Be^{at} = 0$$

$$\rightarrow RCa + 1 = 0, \qquad \therefore a = -1/(RC)$$

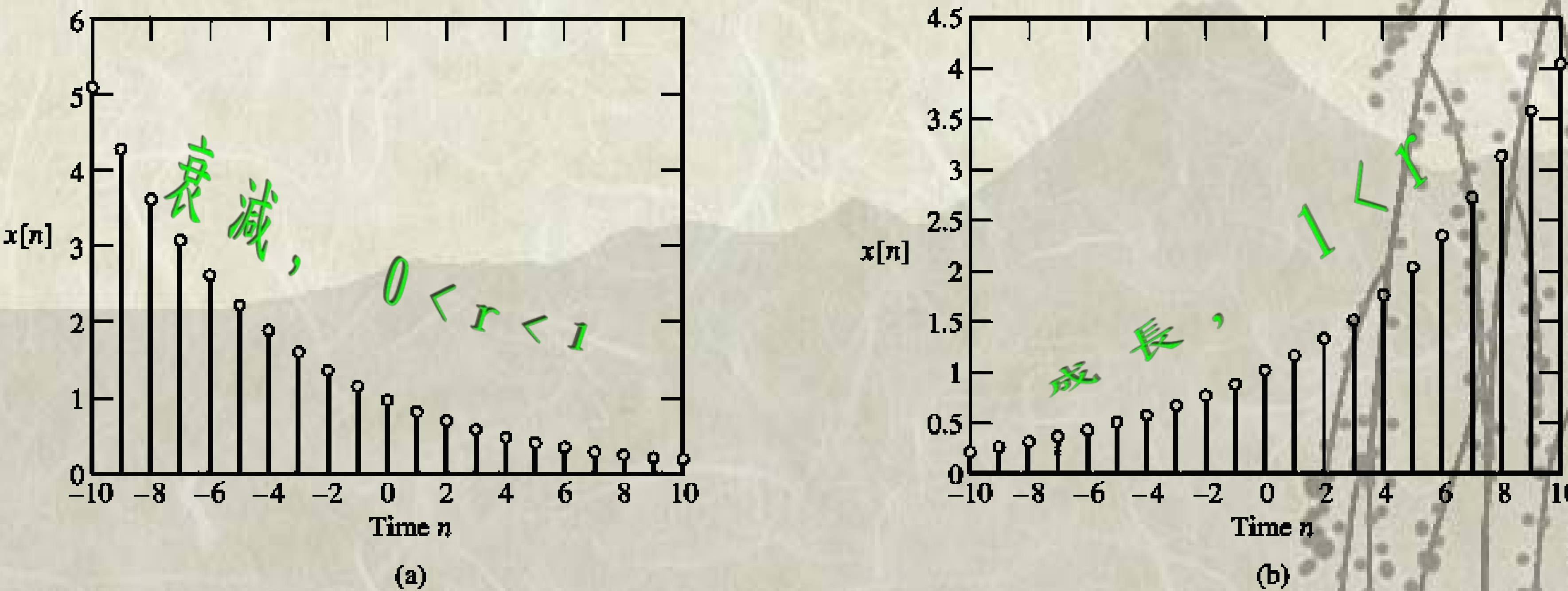
$$\rightarrow v(0) = Be^{a0}, \qquad \therefore B = v(0)$$



$$\Rightarrow v(t) = v(0)e^{-t/(RC)}$$

離散指數訊號(exponential signals)

ee: P35



- (a) $x[n] = Br^n, 0 < r < 1$
- (b) $x[n] = Br^n, r > 1$
- (c) 類比於連續指數 $r = e^{\alpha}$

ee: P36

複數指數訊號

$$x(t) = Be^{at}$$

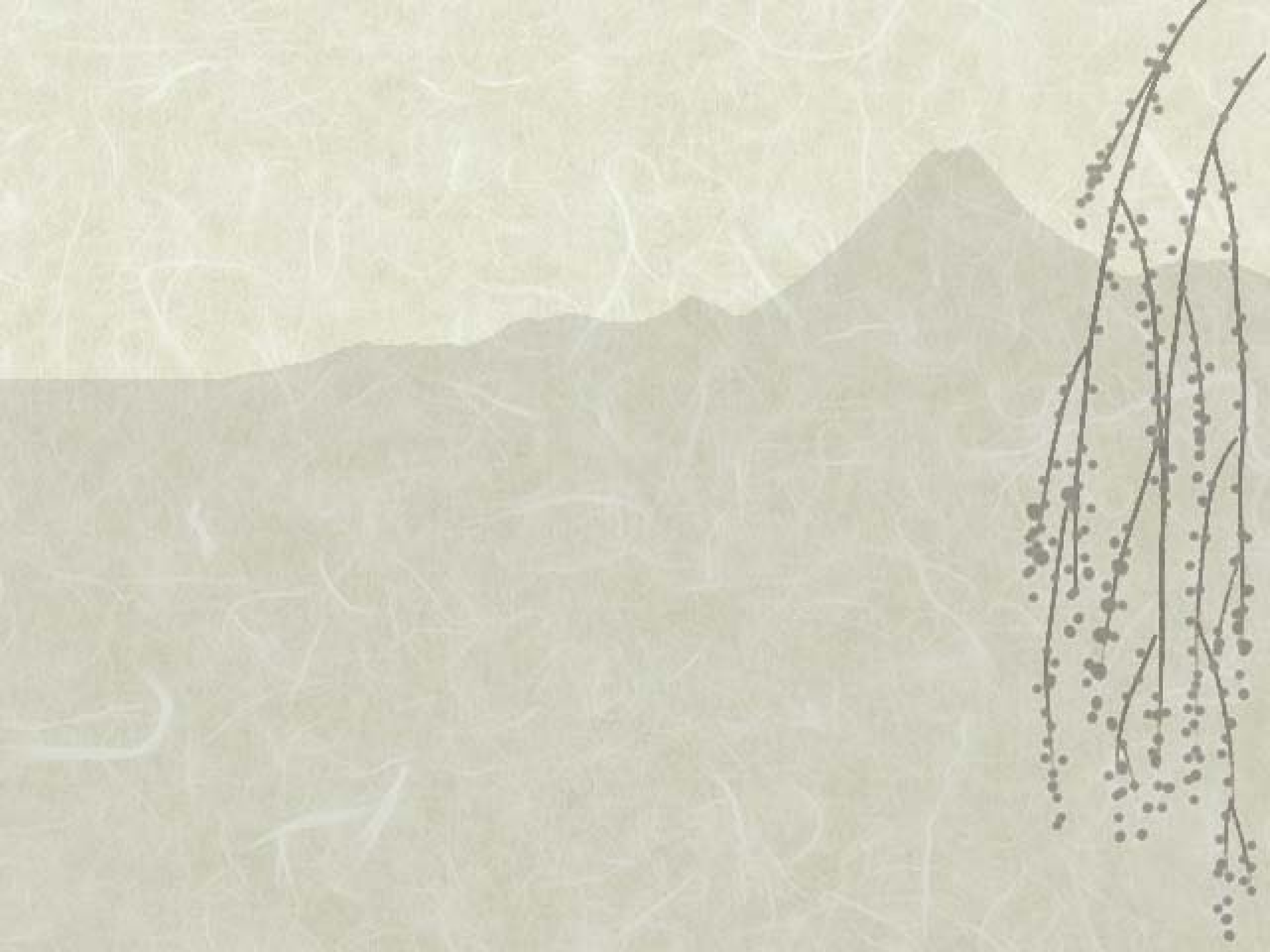
$$x[n] = Br^{n}$$

- 若B, r, a為複數(complex)則可定義出複數指數訊號
- 常見之複數指數訊號

$$x(t) = Be^{j\varpi t}$$

$$x[n] = Be^{j\Omega n} = B\left(e^{j\Omega}\right)^{n}$$

Fourier transform之基本基底訊號
← 連續
← 離散



弦波訊號(sinusoidal signals)

ee: P36

$$x(t) = A\cos(\varpi t + \phi) \qquad (1.35)$$

重要參數

A:振幅

ϖ:角頻率

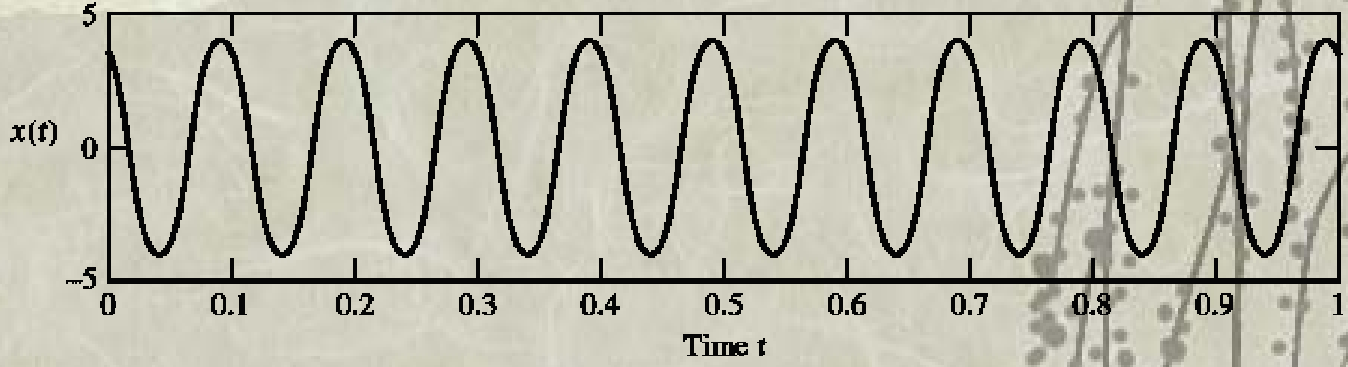
$\varpi = 2\pi f$

ϕ:啟始相位

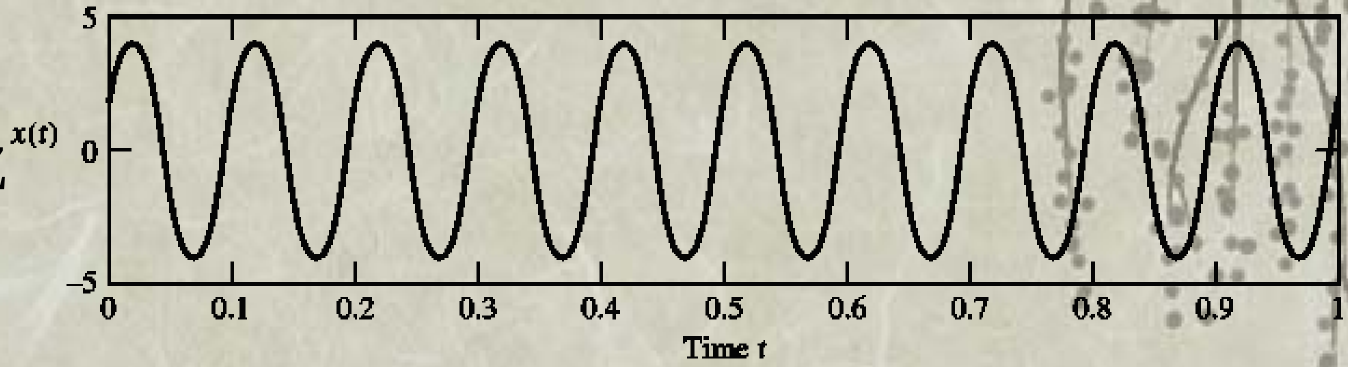
$x(t+T) = x(t)$

$\varpi T = n2\pi, n \in \mathbf{Z}$

$\rightarrow T = \dfrac{2\pi}{\varpi}$



(a)



(b)

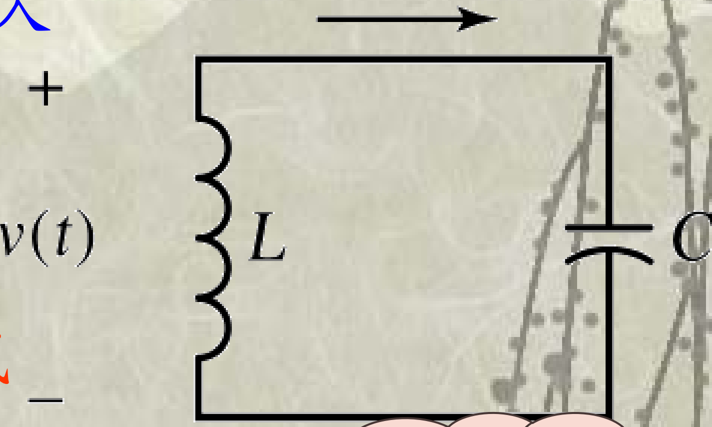
計算: 二階LC 電路

$$v(t)+\frac{1}{C}\int i(t)dt=0$$

$$i(t)=C\frac{d}{dt}v(t)$$

代入

$$L\frac{d}{dt}i(t)+\frac{1}{C}\int i(t)dt=0$$



$$LC\frac{d^2}{dt^2}v(t)+v(t)=0$$

二階常微分方程式

假設$v(t)=Be^{at}$

$$LCa^2Be^{at}+Be^{at}=0$$

振盪電路之自然頻率

$$\rightarrow RCa^2+1=0,\qquad \therefore a=\sqrt{-1/(RL)}=j\varpi_0$$

$$\rightarrow \varpi_0=\frac{1}{\sqrt{LC}},$$

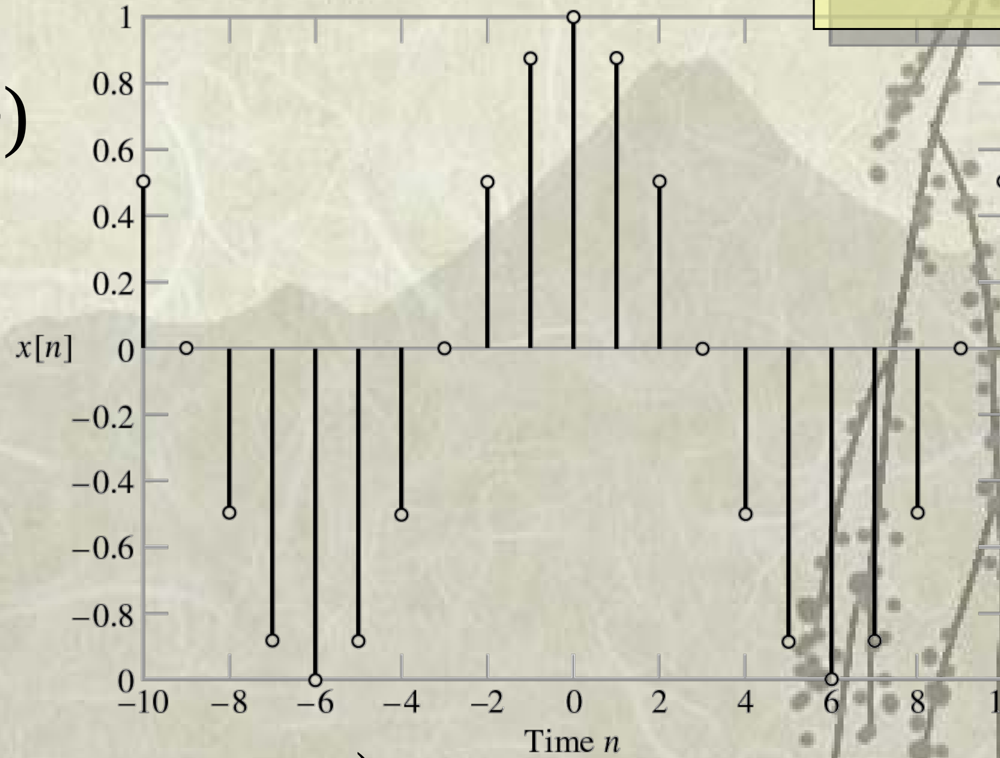
a為複數,解型態$A_r\cos(\varpi_0 t)+jA_i\sin(\varpi_0 t)$

代入初值$v(0), i(0)=0 \Rightarrow v(t)=v(0)\cos(\varpi_0 t)$

離散弦波訊號

ee: P37,38

$$x[n] = A\cos(\Omega n + \phi)$$



$$x[n+N] = A\cos\left(\Omega(n+N) + \phi\right)$$

$$\Rightarrow \Omega N = 2\pi m \qquad radians,$$

$$or \qquad \Omega = \frac{2\pi m}{N} \qquad rad/cycle, \quad m, N \in 整數$$

基本三角恆等式

ee: P763,

A-1

$$\cos^2(\theta) = \tfrac{1}{2}\left(1+\cos(2\theta)\right)$$

$$\sin^2(\theta) = \tfrac{1}{2}\left(1-\cos(2\theta)\right)$$

$$\cos(2\theta) = 2\cos^2(\theta) - 1 = 1 - 2\sin^2(\theta)$$

$$\sin(2\theta) = 2\sin(\theta)\cos(\theta)$$

$$\sin(\theta \pm \phi) = \sin(\theta)\cos(\phi) \pm \cos(\theta)\sin(\phi)$$

$$\cos(\theta \pm \phi) = \cos(\theta)\cos(\phi) \mp \sin(\theta)\sin(\phi)$$

$$\sin(\theta)\sin(\phi) = \tfrac{1}{2}\left[\cos(\theta-\phi) - \cos(\theta+\phi)\right]$$

$$\cos(\theta)\cos(\phi) = \tfrac{1}{2}\left[\cos(\theta-\phi) + \cos(\theta+\phi)\right]$$

$$\sin(\theta)\cos(\phi) = \tfrac{1}{2}\left[\sin(\theta-\phi) + \sin(\theta+\phi)\right]$$

Ex 1.7

ee: P38~39

Ex: 1.7

$$x_1[n]=\sin(5\pi n),\qquad x_2[n]=\sqrt{3}\cos(5\pi n)$$

(a)求主週期(fundamental period)

(b)求$y[n]=x_1[n]+x_2[n]=A\cos(\Omega n+\phi)$

$$\Omega N=2\pi m,\qquad \Omega=5\pi$$

$$N=\frac{2\pi m}{\Omega}\xrightarrow{\text{最小整數}}N=2$$

$$y[n]=x_1[n]+x_2[n]=\sin(5\pi n)+\sqrt{3}\cos(5\pi n)$$

$$y[n]=A\cos(\Omega n+\phi)=A\left[\cos(\Omega n)\cos(\phi)-\sin(\Omega n)\sin(\phi)\right]$$

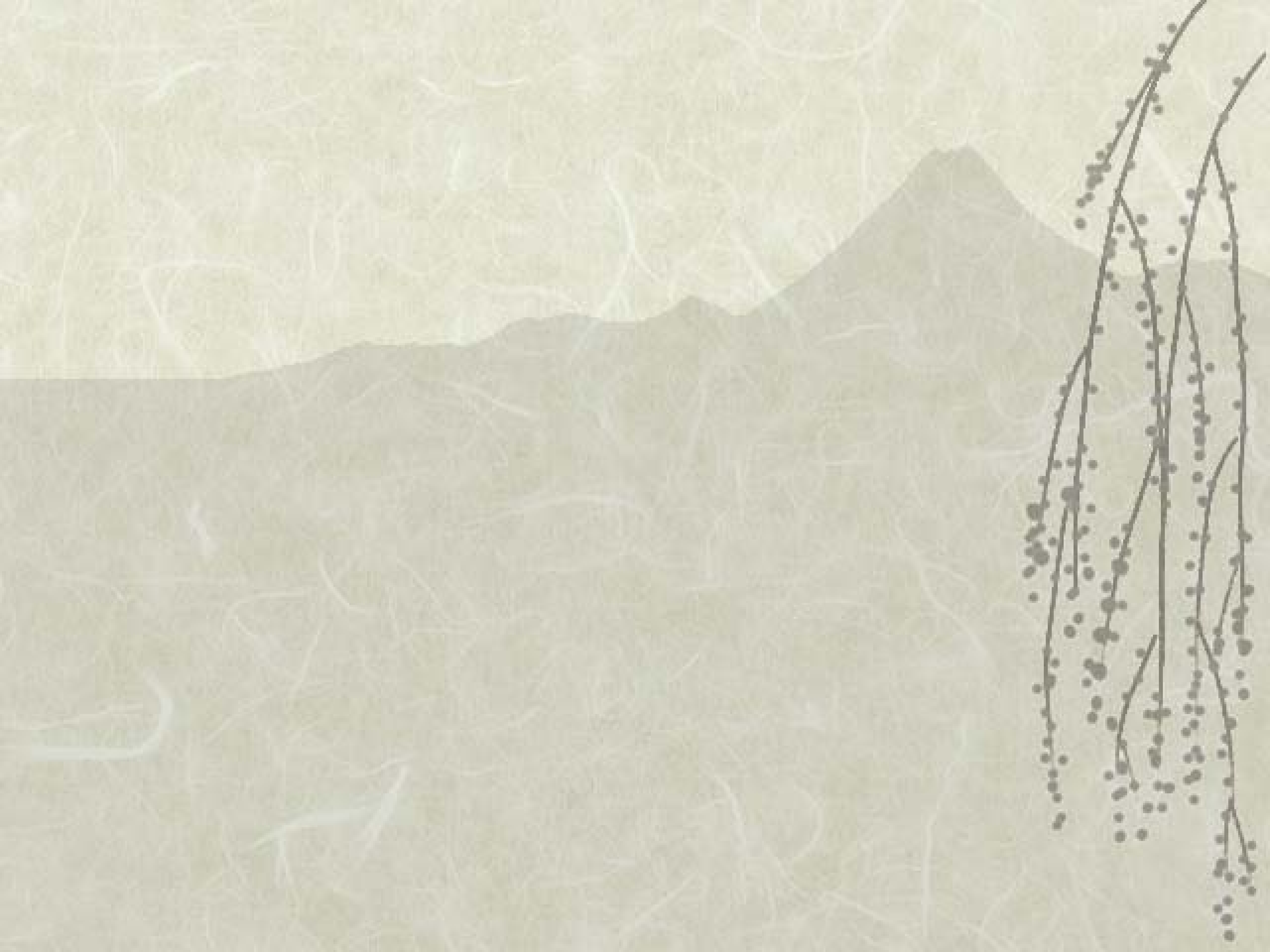
$$\to\Omega=5\pi,$$

$$\to A\sin(\phi)=-1,$$

$$\to A\cos(\phi)=\sqrt{3}$$

$$\to\frac{A\sin(\phi)}{A\cos(\phi)}=\tan(\phi)=\frac{-1}{\sqrt{3}},\therefore\phi=-\pi/3$$

$$A\sin(\phi)=-1\to A=-1/\sin\left(-\pi/3\right)=2$$



弦波訊號與複數指數訊號

$$Euler's\ identity$$

$$e^{j\theta} = \cos(\theta) + j\sin(\theta)$$

複數指數訊號

$$Be^{j\varpi t}, \quad 其中 B = Ae^{j\phi}$$

$$Be^{j\varpi t} = Ae^{j\phi}e^{j\varpi t} = Ae^{j(\varpi t+\phi)}$$

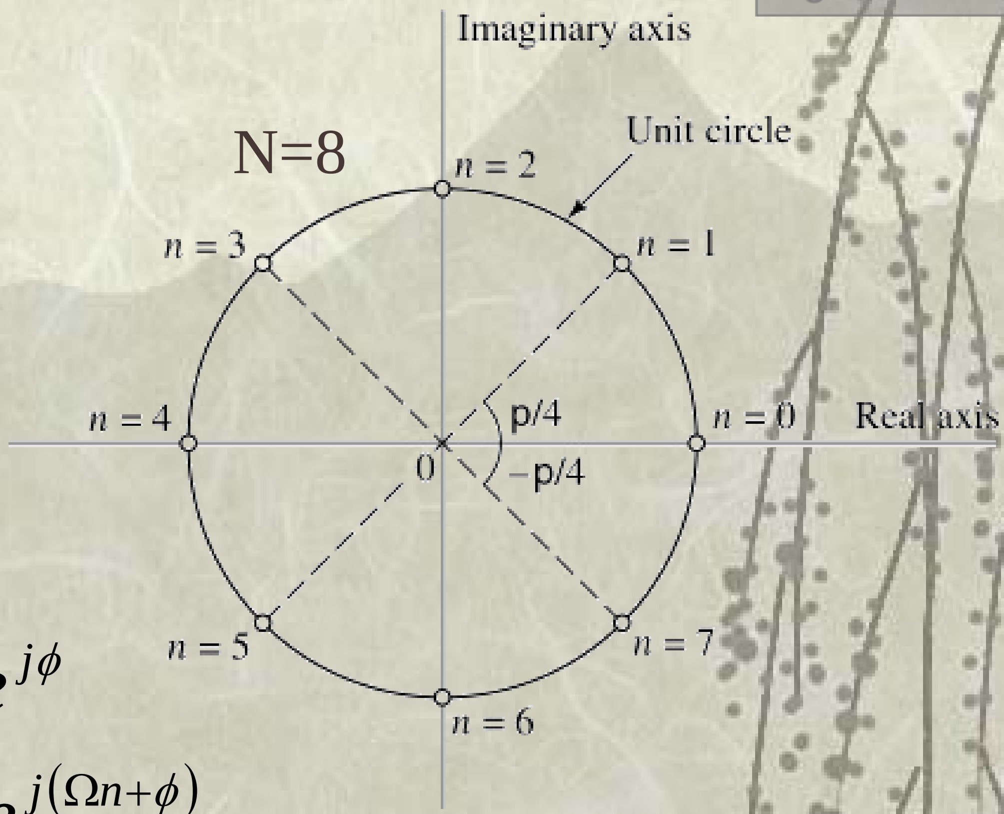
$$\xrightarrow{\text{取實部}} A\cos(\varpi t + \phi) = \mathrm{Re}\{Be^{j\varpi t}\}$$

$$\xrightarrow{\text{取虛部}} A\sin(\varpi t + \phi) = \mathrm{Im}\{Be^{j\varpi t}\}$$

離散弦波訊號與複數指數訊號

ee: P41

Fig 1.34



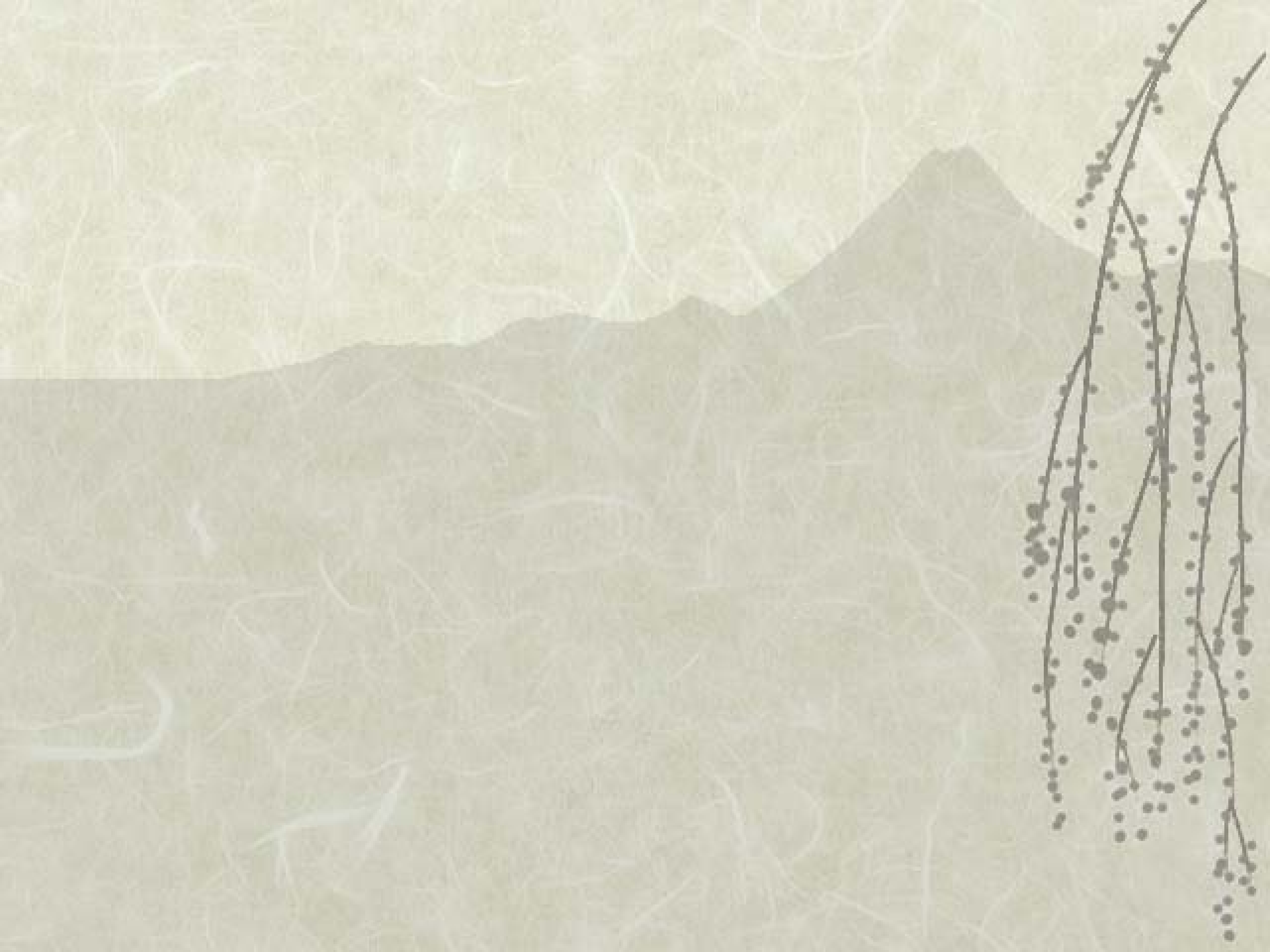
複數指數訊號

$$Be^{j\Omega n}, \quad 其中 B = Ae^{j\phi}$$

$$Be^{j\Omega n} = Ae^{j\phi}e^{j\Omega n} = Ae^{j(\Omega n+\phi)}$$

$$\xrightarrow{取實部} A\cos(\Omega n+\phi) = \mathrm{Re}\{Be^{j\Omega n}\}$$

$$\xrightarrow{取虛部} A\sin(\Omega n+\phi) = \mathrm{Im}\{Be^{j\Omega n}\}$$



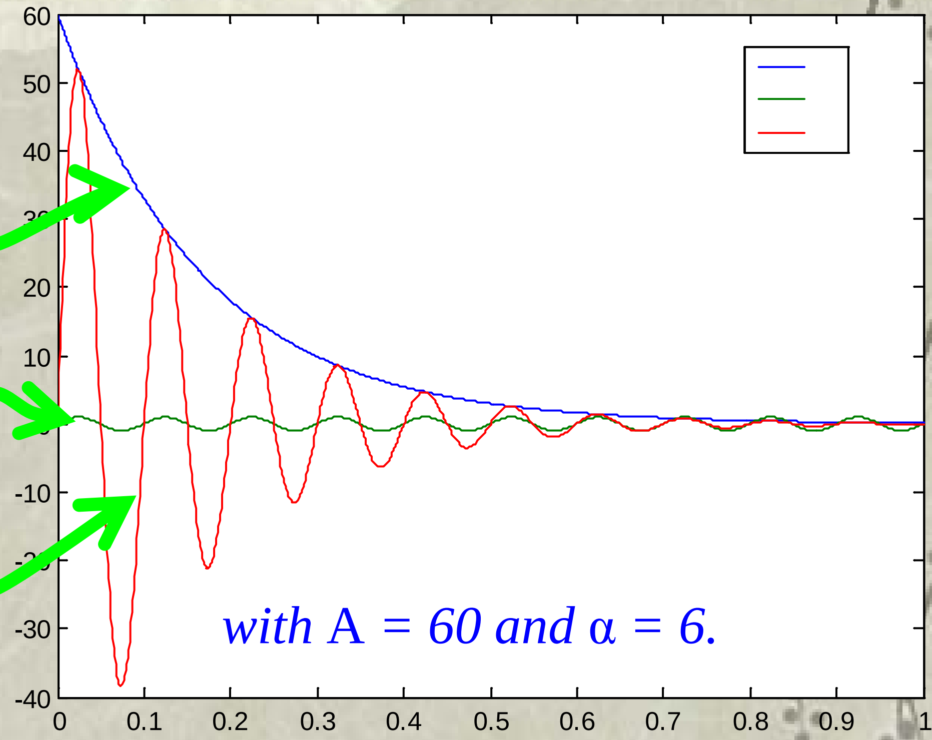
指數衰退之弦波訊號

$$x(t) = Ae^{-at}\sin(\varpi t + \phi)$$

$$x_1(t) = Ae^{-at}$$

$$x_2(t) = \sin(\varpi t + \phi)$$

$$x(t) = x_1(t)x_2(t)$$



計算: RLC 電路

$KCL:$

$$C\frac{d}{dt}v(t)+\frac{1}{R}v(t)+\frac{1}{L}\int_{-\infty}^{t}v(\tau)d\tau=0$$

$$C\frac{d^2}{dt^2}v(t)+\frac{1}{R}\frac{d}{dt}v(t)+\frac{1}{L}v(t)=0$$

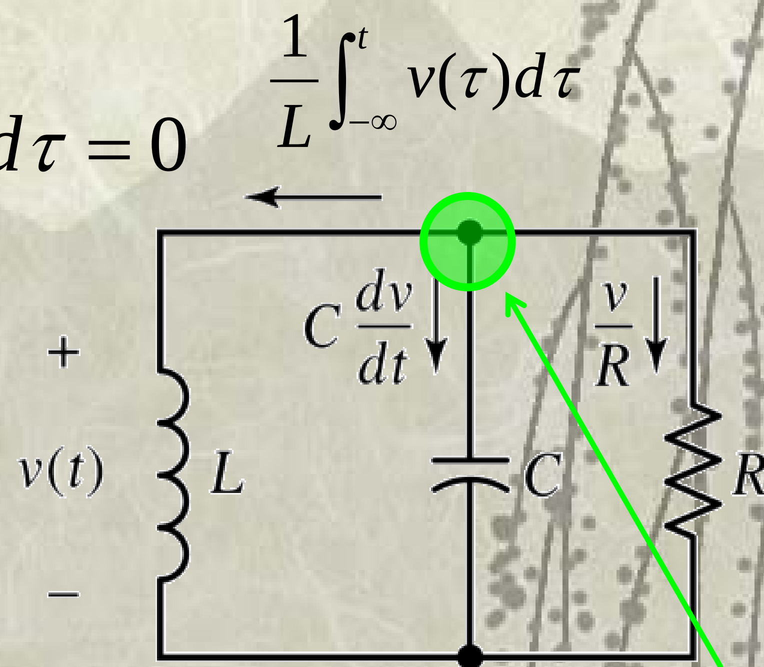
假設解$v(t)=Ae^{(\sigma+j\varpi)t}$

$$RLC(\sigma+j\omega)^2+L(\sigma+j\varpi)+R=0$$

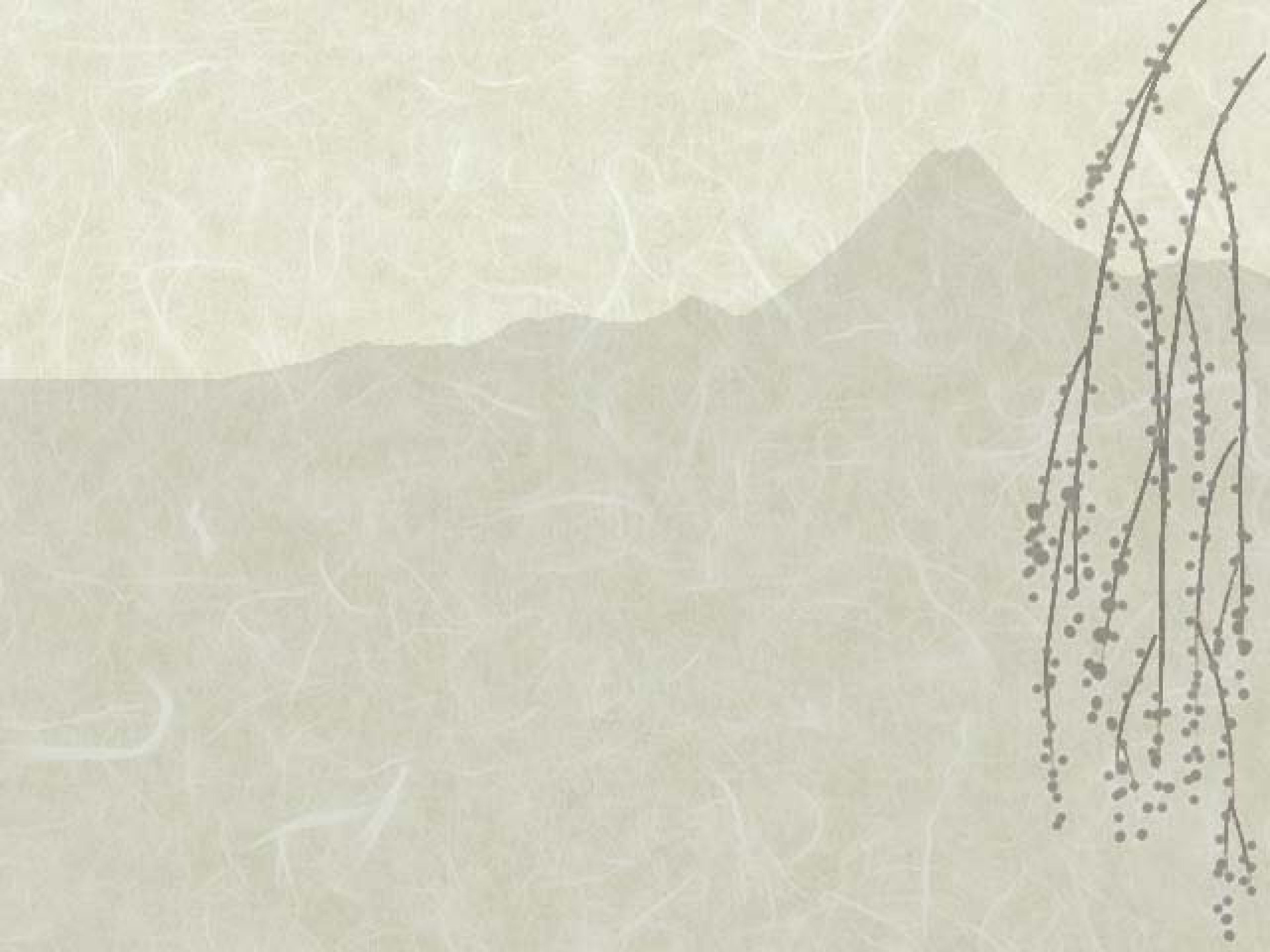
$$\rightarrow\sigma+j\omega=\frac{-L\pm\sqrt{L^2-4R^2LC}}{2RLC}=\frac{-1}{2RC}\pm j\sqrt{\frac{1}{LC}-\frac{1}{4R^2C^2}}$$

$$\therefore\sigma=\frac{-1}{2RC},\qquad\omega=\sqrt{\frac{1}{LC}-\frac{1}{4R^2C^2}}\qquad assume\qquad R>\sqrt{L/(4C)}$$

初值$v(0)=Ae^0=A$



節點總流出電流=0



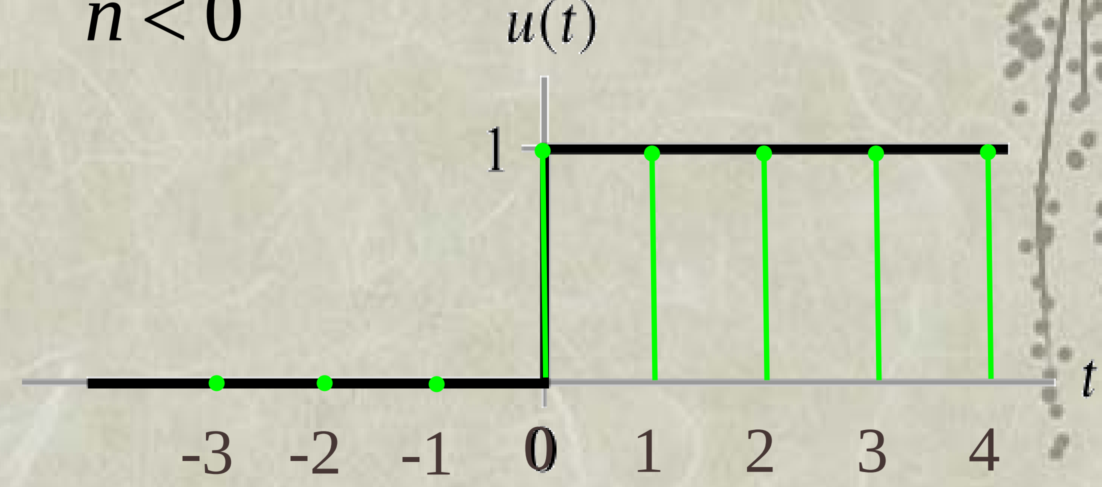
單位步階訊號, unit-step signal

ee: P43

Fig 1.38

$$u(t)=\begin{cases}1, & t\geq 0\\ 0, & t<0\end{cases}$$

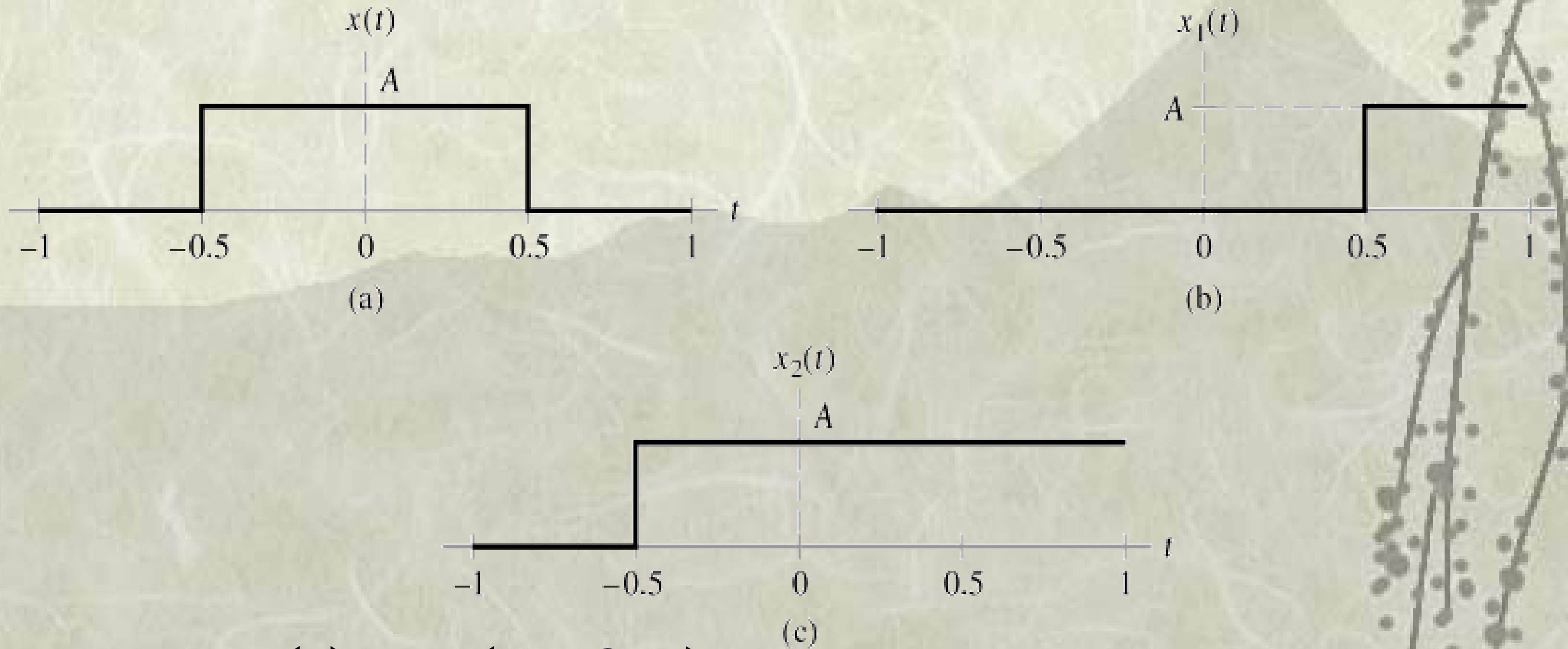
$$u[n]=\begin{cases}1, & n\geq 0\\ 0, & n<0\end{cases}$$



以單位步階訊號表示下列訊號

ee: P44

Fig 1.39



$$x_1(t) = u(t - 0.5)$$

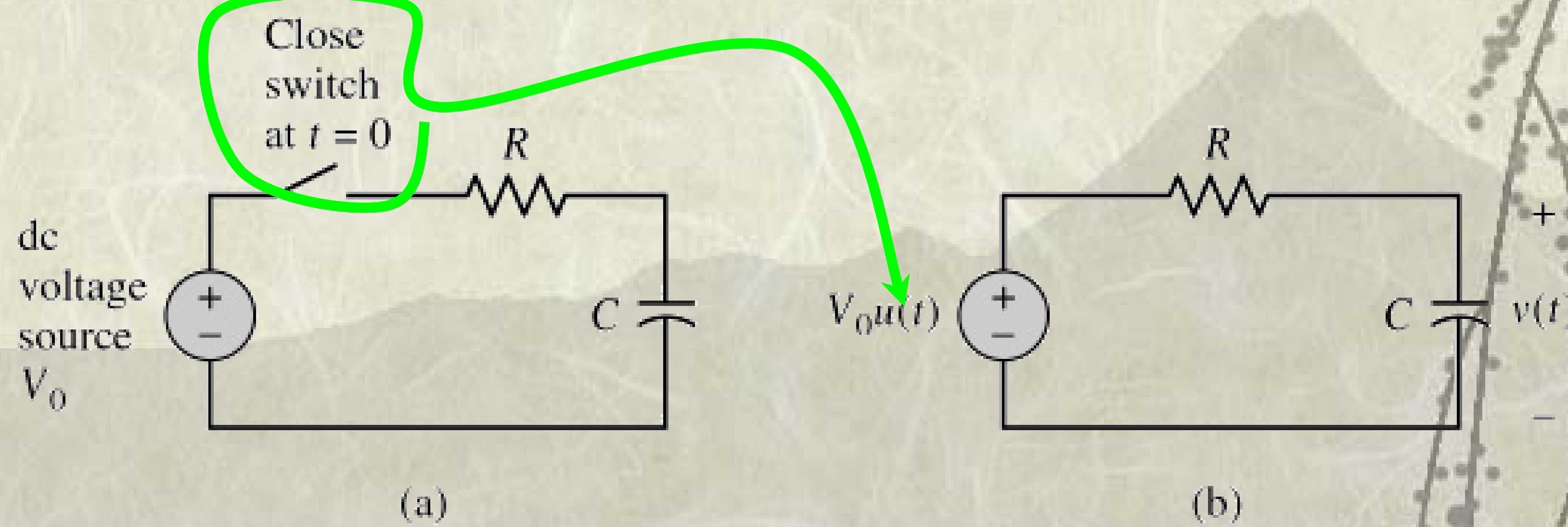
$$x_2(t) = u(t + 0.5) = u(t - (-0.5))$$

$$\therefore x(t) = x_2(t) - x_1(t) = u(t + 0.5) - u(t - 0.5)$$

以步階訊號描述開關動作

ee: P45

Fig 1.40



$$v(t) = v_P + Be^{at}, \qquad v_p \text{為特解}$$

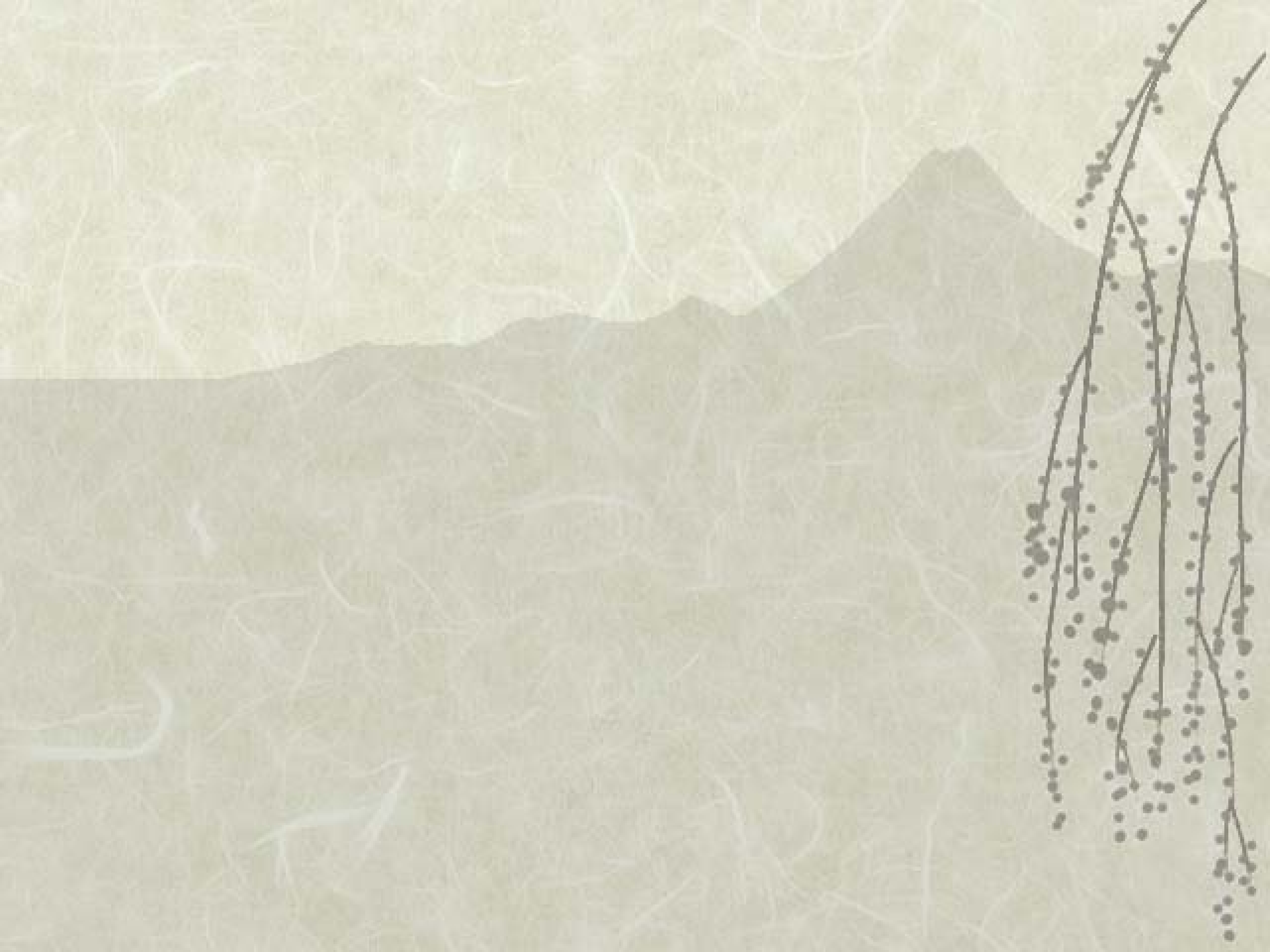
$$a = -1/(RC)$$

$$\because v(0) = 0, v(\infty) = V_0$$

$$\rightarrow v_p + B = 0, \qquad V_0 = v_p \rightarrow B = -v_p$$

$$v(t) = V_0(1 - e^{-t/(RC)})u(t)$$

描述開關動作後之輸出

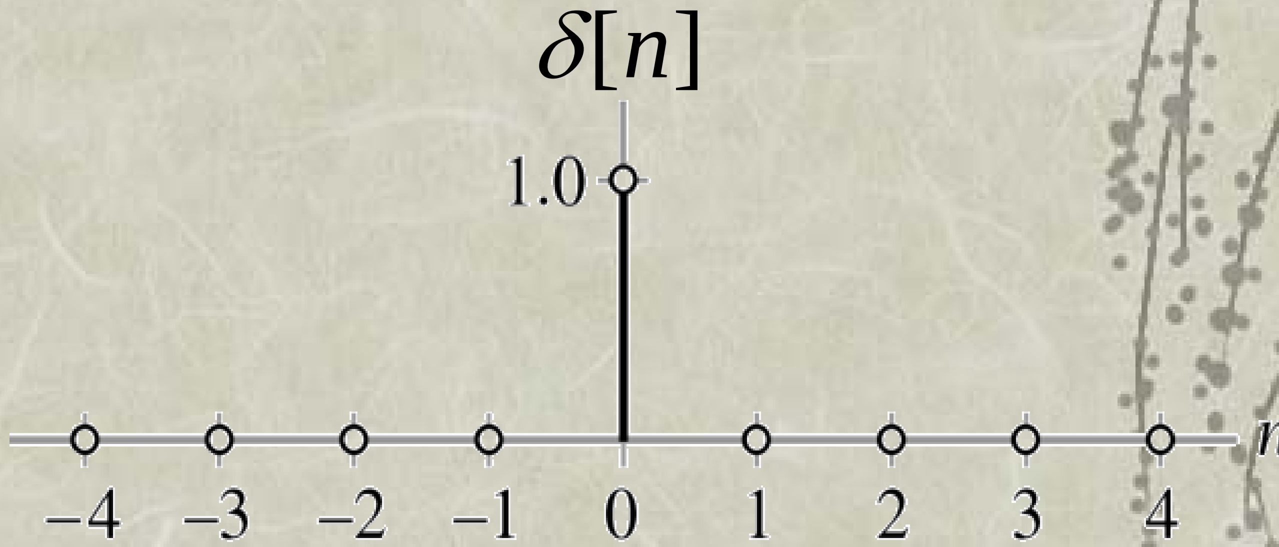


離散脈衝(Discrete-time form of impulse.)

ee: P46

Fig 1.41

$$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$



脈衝(Dirac delta)

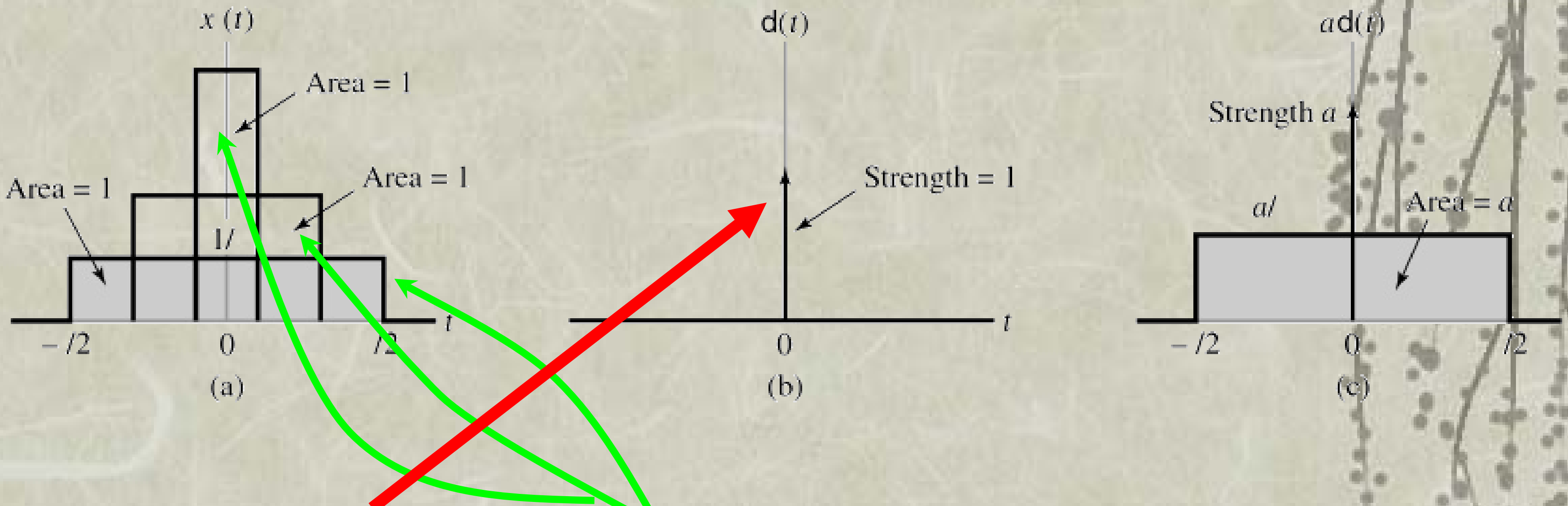
ee: P46

Fig 1.42

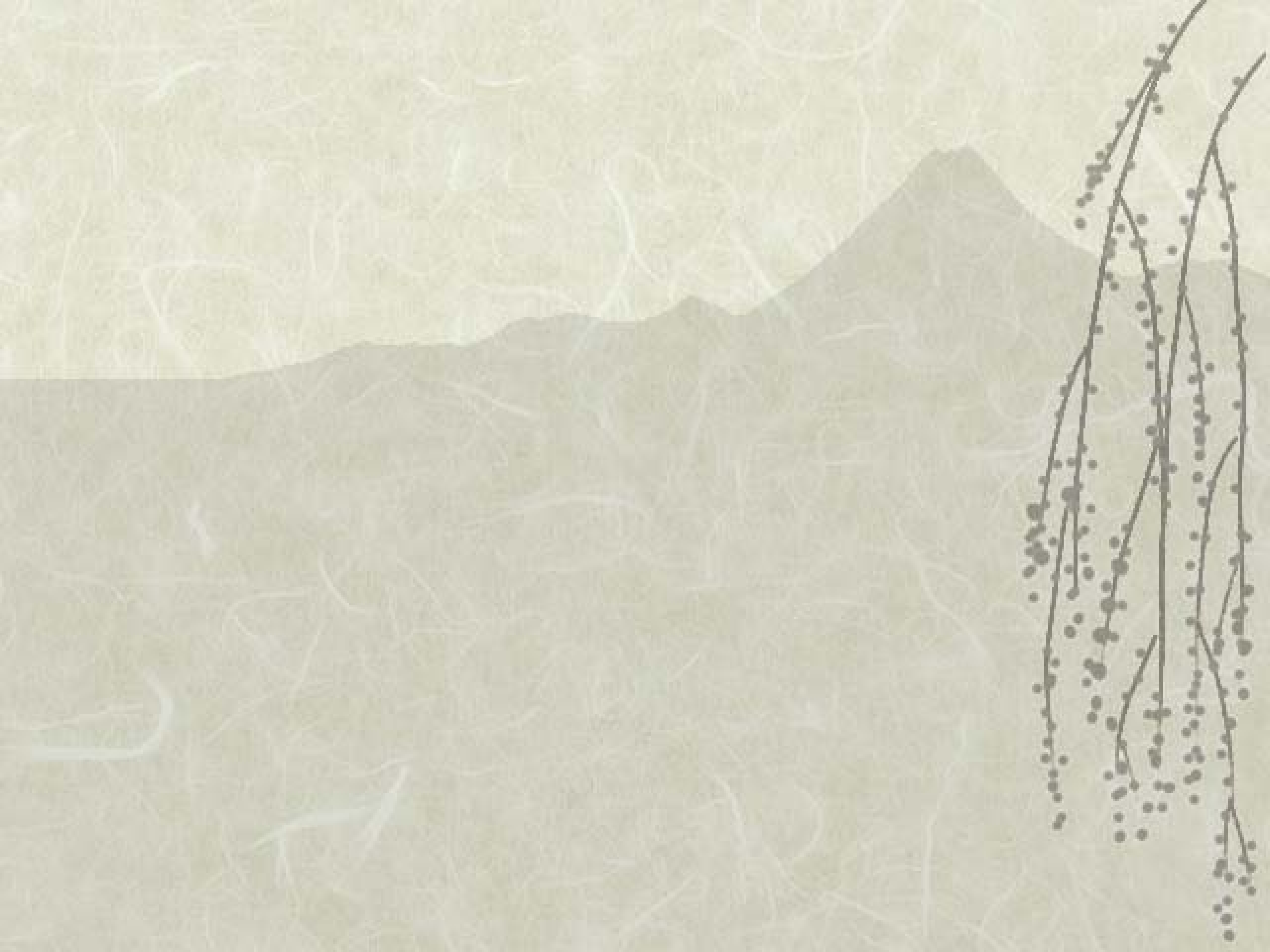
$$\delta(t) = 0, \qquad for \;\; t \neq 0$$

and

$$\int_{-\infty}^{\infty} \delta(t)dt = 1$$



$$\delta(t) = \lim_{\Delta \to 0} x_{\Delta}(t)$$



Dirac delta & unit step

$$\delta(t) = \frac{d}{dt}u(t)$$

$$u(t) = \int_{-\infty}^{t} \delta(\tau)d\tau$$

$$\delta(t) = 0, \qquad for \;\; t \neq 0$$

and

$$\int_{-\infty}^{\infty} \delta(t)dt = 1$$

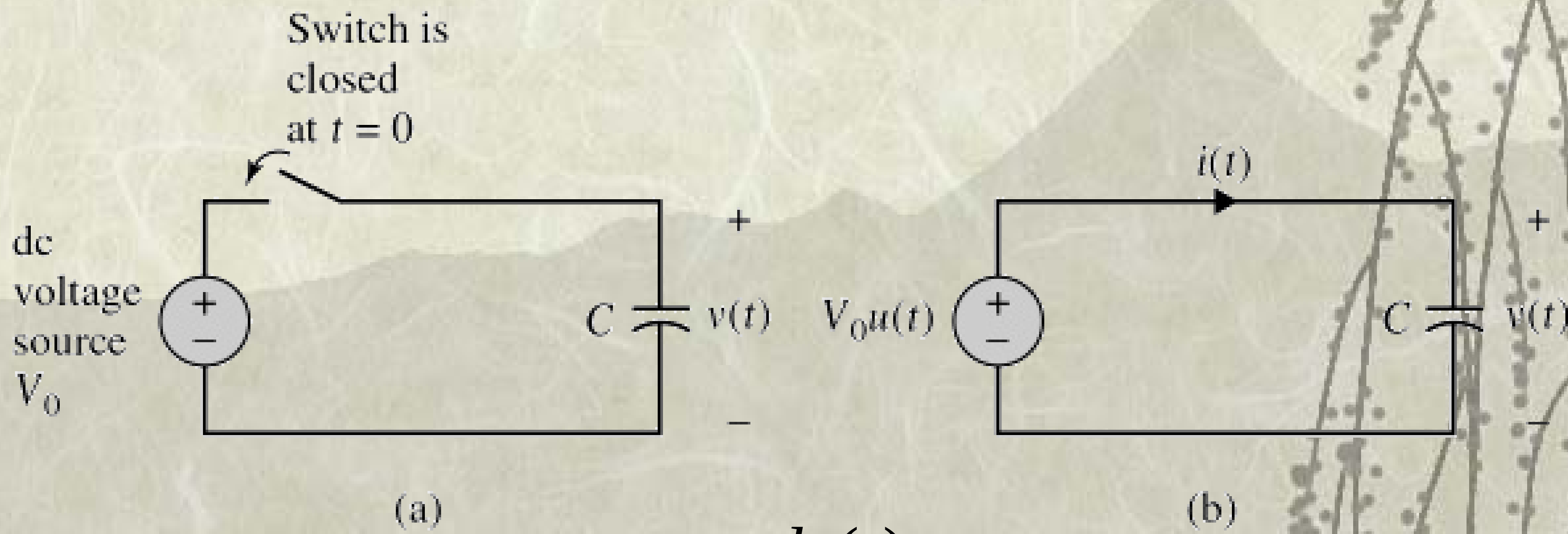
⇔

$$u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

Ex 1.10

ee: P47

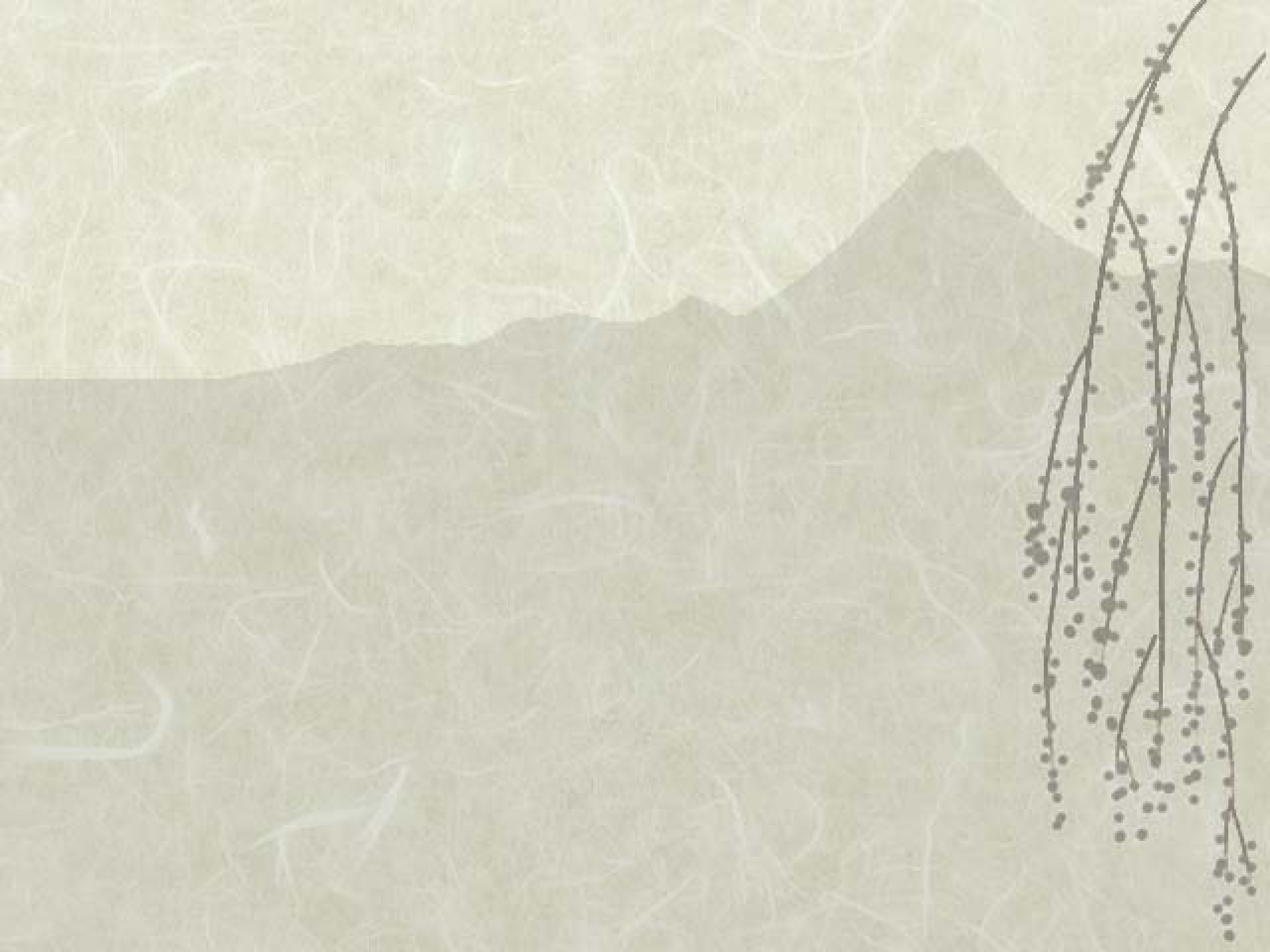
Fig 1.43



$$i(t) = C\frac{dv(t)}{dt}$$

$$\because v(t) = V_0 u(t)$$

$$\therefore i(t) = CV_0 \delta(t)$$



Dirac delta之時間變數運算

ee: P48

$shifting$ & 積分 :

$$\int_{-\infty}^{\infty} x(t)\delta(t-t_0) = x(t-t_0)$$

Dirac delta 可視 為 一 shift operator

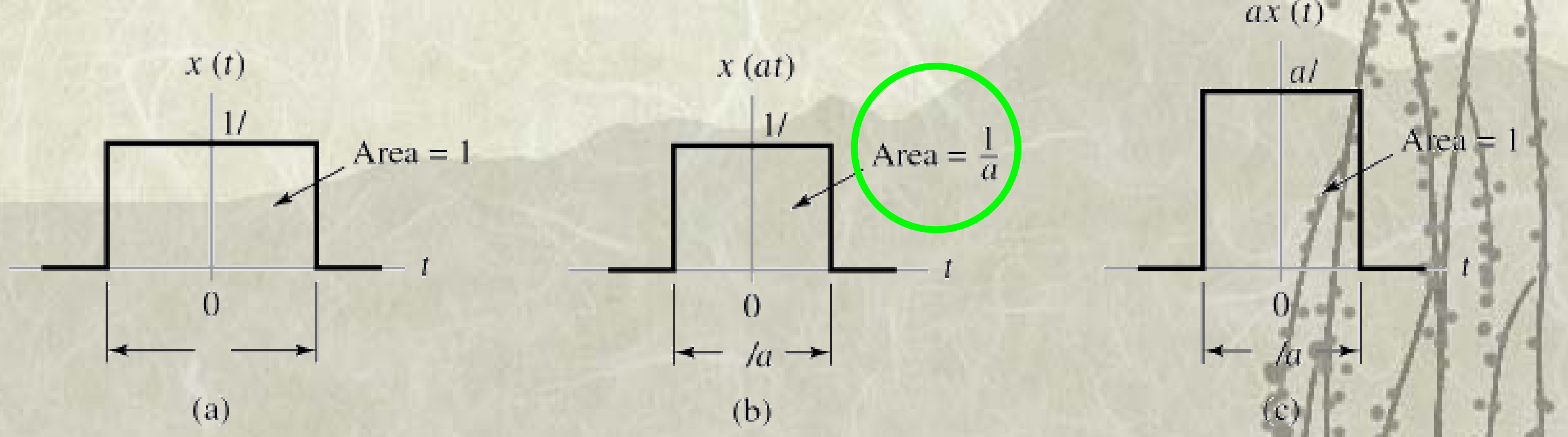
$scaling$:

$$\delta(at) = \frac{1}{a}\delta(t), a > 0$$

圖解脈衝訊號之scaling 運算

ee: P48

Fig. 1.44



$$\delta(at) = \lim_{\Delta\to 0} x_\Delta(at) = \lim_{\Delta\to 0} \frac{1}{a} x_\Delta(t)$$

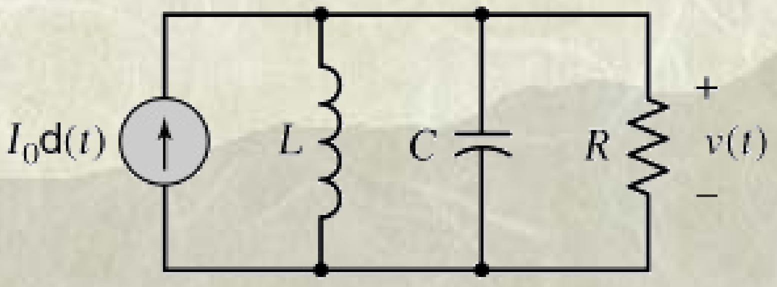
$$\delta(at) = \frac{1}{a}\delta(t)$$

Parallel LRC

ee: P49

Fig. 1.45

(a) find the value of the voltage $v(t)$ at $t = 0^+$



(a)

$$d(t) = \delta(t)$$

$$(a)\ \ I_0\delta(t) = \frac{1}{L}\int_{-\infty}^{t} v(t)dt + C\frac{d}{dt}v(t) + \frac{1}{R}v(t)$$

$$\xrightarrow{\text{兩邊積分至}0^+} \int_{-\infty}^{0^+} I_0\delta(t) = Cv(0^+) \qquad ,\therefore V_0 = I_0 / C$$

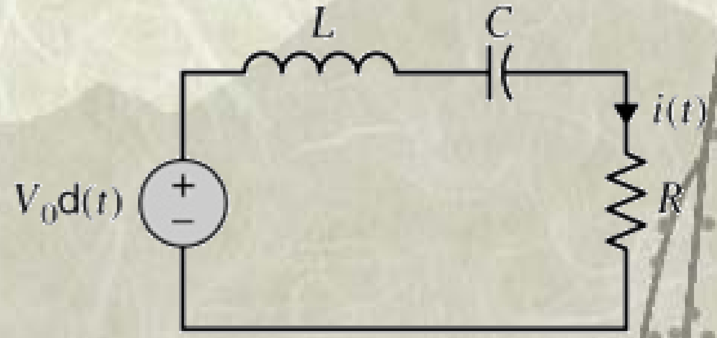
P1.23 series LCR

ee: P49

Fig. 1.45

(a) find the value of the current $i(t)$ at $t = 0^+$

(b) write the integro-differential equation of $i(t)$ for $t \geq 0^+$



$$d(t) = \delta(t)$$

$$(a)\ V_0\delta(t) = L\frac{d}{dt}i(t) + \frac{1}{C}\int_{-\infty}^{t} i(\tau)d\tau + Ri(t)$$

$$\xrightarrow{\text{兩邊積分至}0^+} \int_{-\infty}^{0^+} V_0\delta(t) = Li(0^+) \qquad ,\therefore I_0 = V_0 / L$$

$$(b)\ V_0\delta(t) = L\frac{d}{dt}i(t) + \frac{1}{C}\int_{-\infty}^{t} i(\tau)d\tau + Ri(t)$$

$$\xrightarrow{t \geq 0^+} L\frac{d}{dt}i(t) + \frac{1}{C}\int_{-\infty}^{t} i(\tau)d\tau + Ri(t) = 0$$

ee: P50

Derivatives of the Impulse

$$\delta^{(1)}(t)=\frac{d\delta(t)}{dt}=\lim_{\Delta\to 0}\frac{1}{\Delta}\left[\delta\left(t+\Delta/2\right)-\delta\left(t-\Delta/2\right)\right]$$

*doublet*性質

$$\int_{-\infty}^{\infty}\delta^{(1)}(t)dt=0 \Rightarrow =\lim_{\Delta\to 0}\frac{1}{\Delta}\left[\underbrace{\int_{-\infty}^{\infty}\delta\left(t+\Delta/2\right)dt}_{=1}-\underbrace{\int_{-\infty}^{\infty}\delta\left(t-\Delta/2\right)dt}_{=1}\right]=0$$

$$\int_{-\infty}^{\infty}f(t)\delta^{(1)}(t-t_0)dt=\frac{d}{dt}f(t)\bigg|_{t=t_0}$$

$$=\int_{-\infty}^{\infty}f(t)\lim_{\Delta\to 0}\frac{1}{\Delta}\left[\delta\left(t-t_0+\Delta/2\right)-\delta\left(t-t_0-\Delta/2\right)\right]dt$$

$$=\lim_{\Delta\to 0}\frac{1}{\Delta}\left[f\left(t-t_0+\Delta/2\right)-f\left(t-t_0-\Delta/2\right)\right]=\frac{d}{dt}f(t)\bigg|_{t=t_0}$$

*doublet*微分

$$\frac{d}{dt}\delta^{(1)}(t)=\frac{d^2}{dt^2}\delta(t)$$

$$=\lim_{\Delta\to 0}\frac{\delta^{(1)}\left(t+\Delta/2\right)-\delta^{(1)}\left(t-\Delta/2\right)}{\Delta}$$

ee: P51

P1.24

$$\delta^{(2)}(t)=\frac{d}{dt}\delta^{(1)}(t)=\lim_{\Delta\to 0}\frac{\delta^{(1)}\left(t+\Delta/2\right)-\delta^{(1)}\left(t-\Delta/2\right)}{\Delta}$$

$\delta^{(2)}(t)$性質

(a) $\int_{-\infty}^{\infty} f(t)\delta^{(2)}(t-t_0)dt$

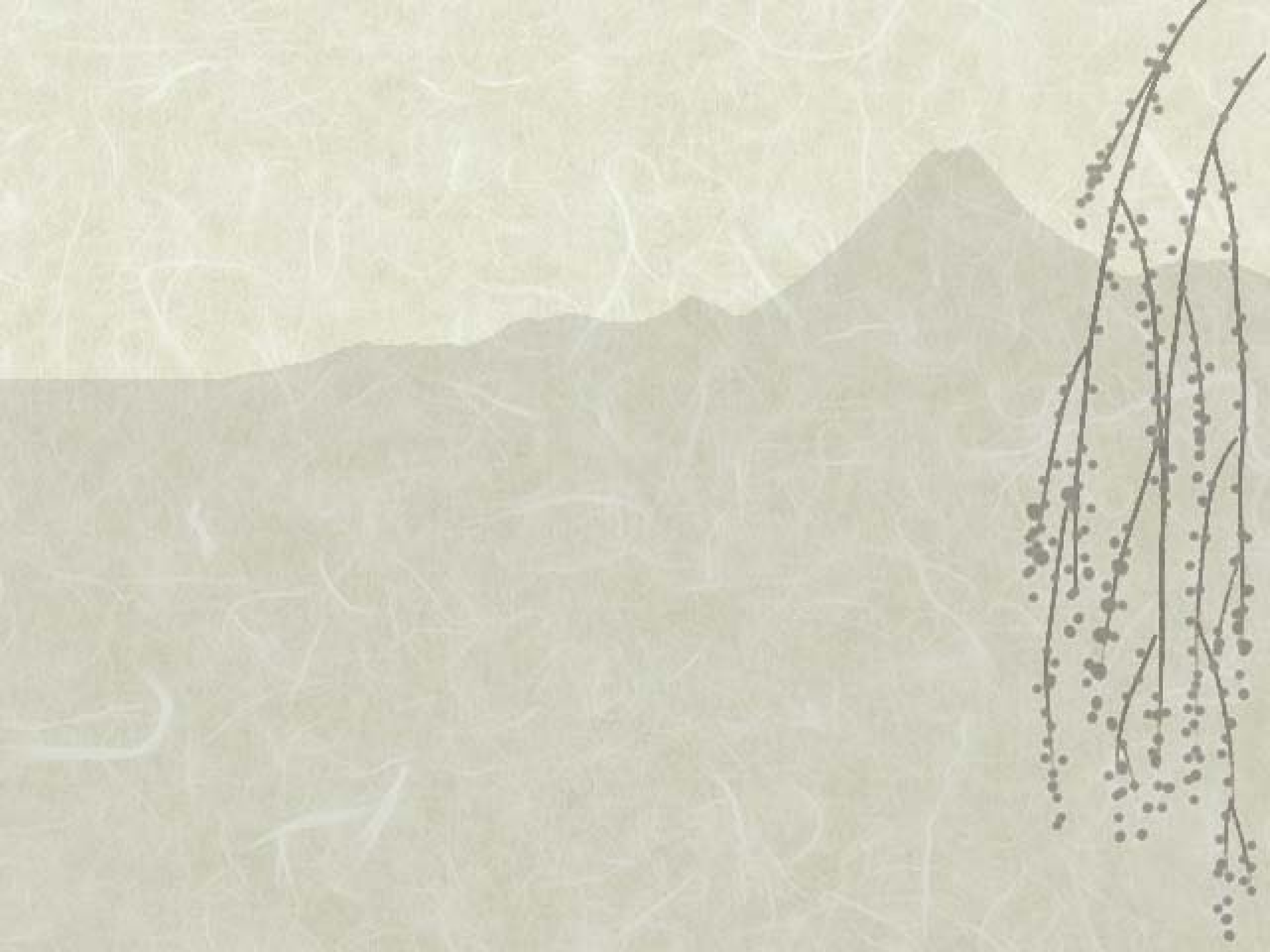
$$=\lim_{\Delta\to 0}\frac{\int_{-\infty}^{\infty} f(t)\delta^{(1)}\left(t-t_0+\Delta/2\right)dt-\int_{-\infty}^{\infty} f(t)\delta^{(1)}\left(t-t_0-\Delta/2\right)dt}{\Delta}$$

$$=\lim_{\Delta\to 0}\frac{\frac{d}{d\tau}f(\tau-t_0)\Big|_{\tau=t+\Delta/2}-\frac{d}{d\tau}f(\tau-t_0)\Big|_{\tau=t-\Delta/2}}{\Delta}=\frac{d}{dt}\left(\frac{d}{dt}f(t)\right)\Bigg|_{t=t_0}$$

(b) $\int_{-\infty}^{\infty} f(t)\delta^{(n)}(t-t_0)dt$

$$=\lim_{\Delta\to 0}\frac{\int_{-\infty}^{\infty} f(t)\delta^{(n-1)}\left(t-t_0+\Delta/2\right)dt-\int_{-\infty}^{\infty} f(t)\delta^{(n-1)}\left(t-t_0-\Delta/2\right)dt}{\Delta}$$

$$=\ldots=\frac{d^n}{dt^n}f(t)\Bigg|_{t=t_0}$$

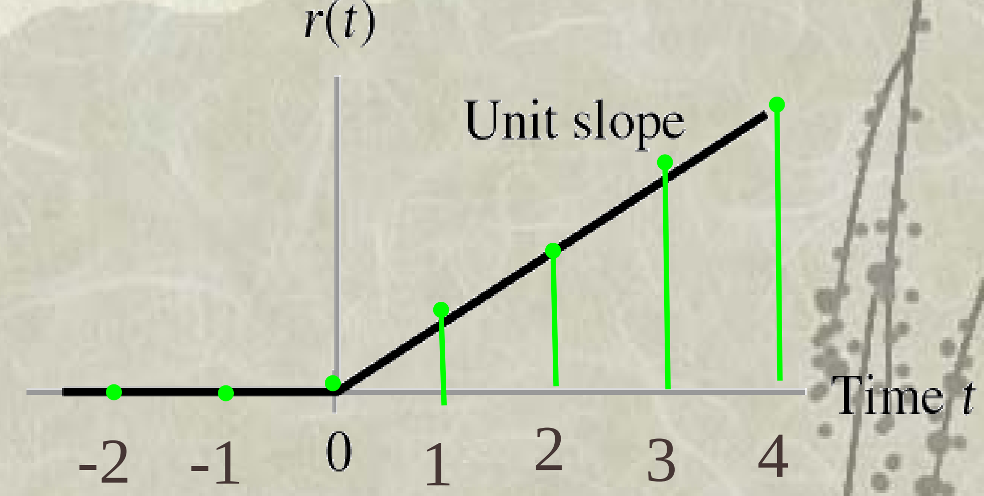


Ramp Signal

ee: P51,52

Fig. 1.46,47

$$r(t)=\begin{cases} t, & t\geq 0 \\ 0, & t<0 \end{cases} = tu(t)$$



$$r[n]=\begin{cases} n, & n\geq 0 \\ 0, & n<0 \end{cases} = nu[n]$$

Ex 1.11

ee: P52

Fig 1.48

$$i(t) = I_0 u(t)$$

$$v(t) = \frac{1}{C}\int_{-\infty}^{t} i(\tau)d\tau$$

$$= \frac{1}{C}\int_{-\infty}^{t} I_0 u(\tau)d\tau$$

$$= \frac{I_0}{C} r(t)$$