單元二
線性非時變系統之時域表示
Time-Domain Representations of Linear Time-Invariance Systems

教學目標

- 瞭解 線性非時變系統(LTI)之
  - Convolution 與LTI system
  - Convolution 之計算
  - 基本性能
  - 表示法

本單元進度

- Convolution Sum
  - Convolution Sum Evaluation procedure
- Convolution integral
  - Convolution integral Evaluation procedure
- LTI system
  - LTI←→impulse response
  - LTI ←→step response
- LTI representation
  - Differential & difference
  - Block diagram
  - State variable description

下單元進度

- Fourier representation of signal and LTI system
- Fourier analysis
  - Property
  - Fourier ←→ signal operation
  - Fourier ←→ system operation

The Convolution Sum

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = x[n] \otimes h[n] = x[n] * h[n]$$

若有一LTI系統

$y[n] = H\{x[n]\}$

且 $h[n] = H\{\delta[n]\}$

則$y[n]$可表示如下

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

$\Rightarrow$

LTI系統

$y[n] = H\{x[n]\},$

且$h[n] = H\{\delta[n]\}$

則$y[n]$可表示如下

$$y[n] = x[n] * h[n] = h[n] * x[n]$$

訊號=脈衝訊號之線性組合

Graphical example illustrating the representation of a signal x[n] as a weighted sum of time-shifted impulses.

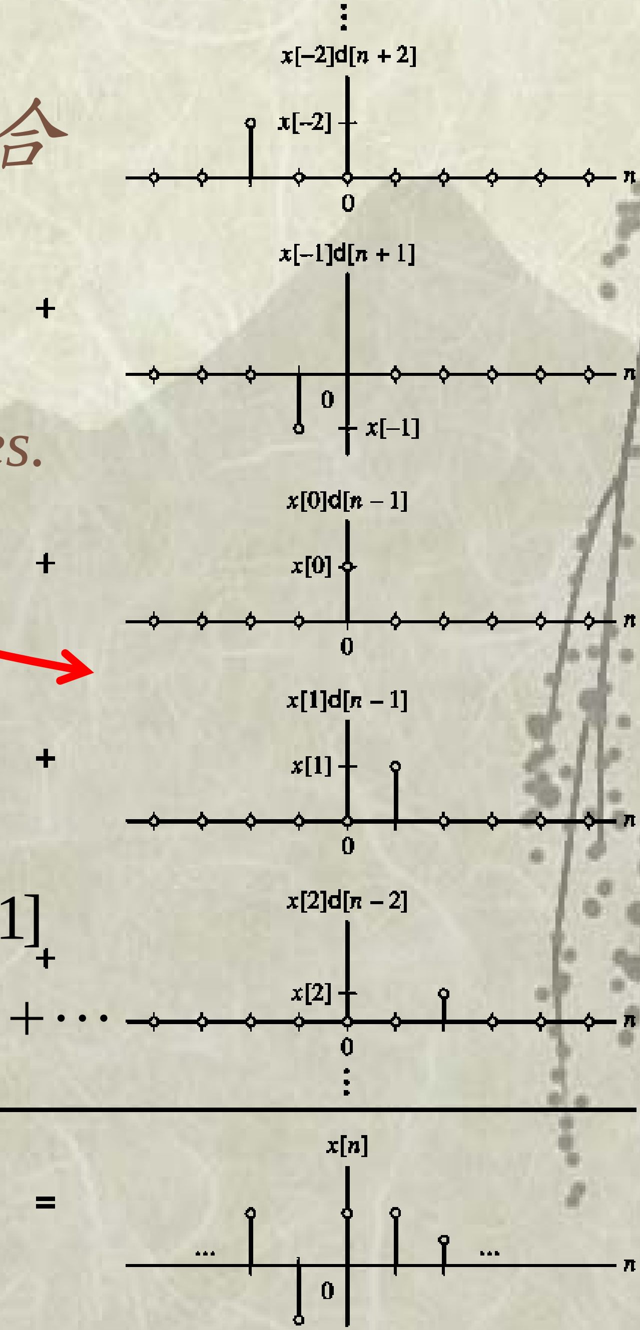
$$x[n]\delta[n] = x[0]\delta[n]$$

shifted $\delta[n-k]$

$$x[n]\delta[n-k] = x[k]\delta[n-k]$$

$$x[n] = \cdots + x[-2]\delta[n+2] + x[-1]\delta[n+1] + x[0]\delta[n] + x[1]\delta[n-1] + x[2]\delta[n-2] + \cdots$$

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$



系統之脈衝響應(impulse response)

假設系統運算H,輸入$x[n]$

$$y[n]=H\left\{x[n]\right\}$$

$$=H\left\{\sum_{k=-\infty}^{\infty}x[k]\delta[n-k]\right\}$$

$$\xrightarrow{H\text{為線性}}y[n]=\sum_{k=-\infty}^{\infty}x[k]H\left\{\delta[n-k]\right\}$$

令$h[n]=H\left\{\delta[n]\right\}\xrightarrow{H\text{為非時變}}h[n-k]=H\left\{\delta[n-k]\right\}$

假設系統運算H,輸入$x[n]$

$$y[n]=H\left\{x[n]\right\}$$

$$\xrightarrow{H\text{ is }LTI\text{ system}}=\sum_{k=-\infty}^{\infty}x[k]h[n-k]=x[n]*h[n]$$

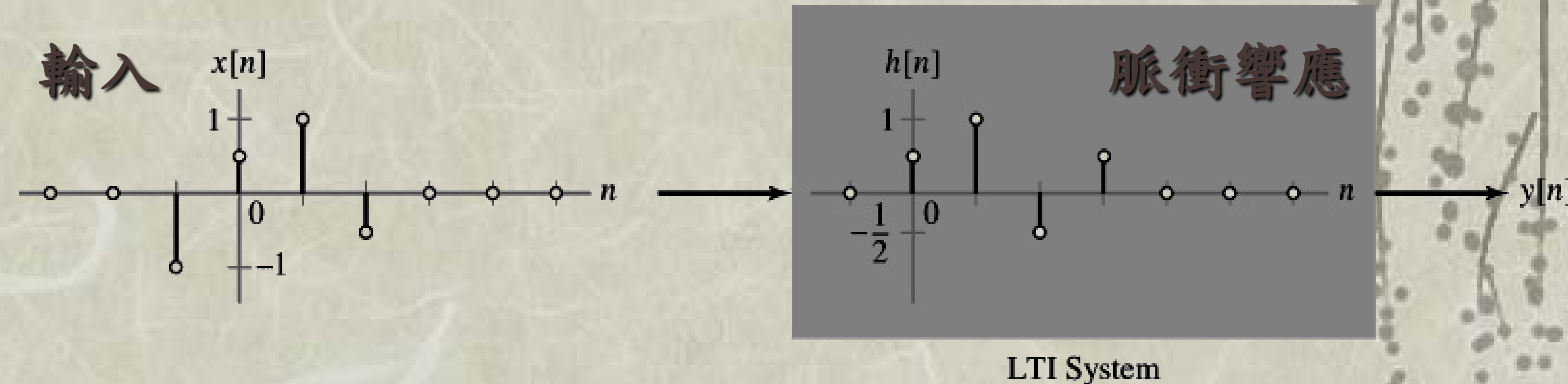
圖解 迴旋和(Convolution Sum)運算

LTI系統, $y[n] = H\{x[n]\}$,

且$h[n] = H\{\delta[n]\}$

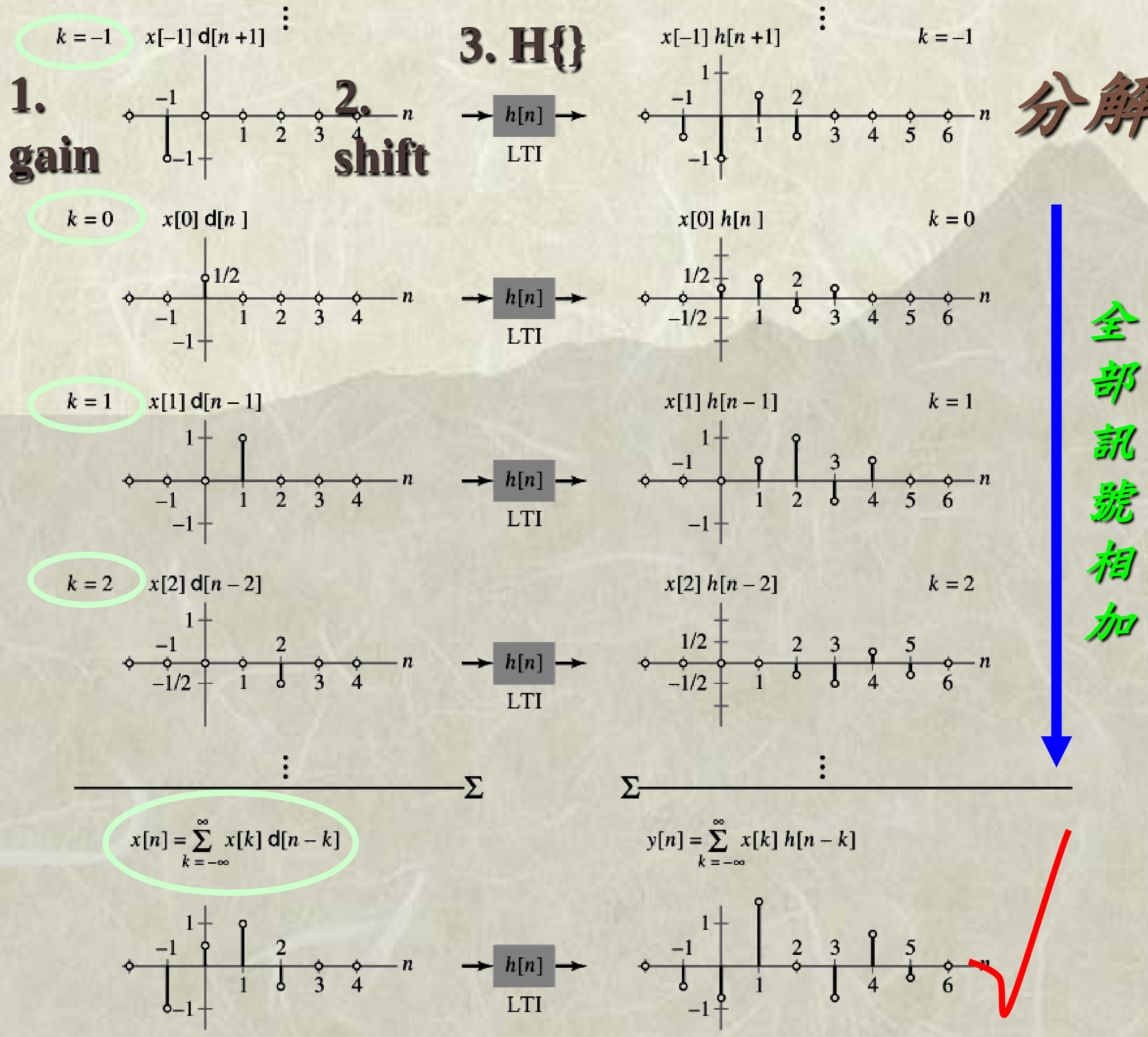
則$y[n]$可表示如下

$$y[n] = x[n] * h[n] = \sum_{-\infty}^{\infty} x[k]h[n-k] = h[n] * x[n]$$



(a)

分解圖示


1. gain
2. shift
3. H{}
k = –1
x[–1] d[n +1]
x[–1] h[n +1]
k = 0
x[0] d[n]
x[0] h[n]
k = 1
x[1] d[n – 1]
x[1] h[n – 1]
k = 2
x[2] d[n – 2]
x[2] h[n – 2]
h[n]
LTI
Σ
x[n] = Σ_{k=–∞}^{∞} x[k] d[n – k]
y[n] = Σ_{k=–∞}^{∞} x[k] h[n – k]
全部訊號相加

Ex 2.1

$$x[n]=\begin{cases}2, & n=0\\ 4, & n=1\\ -2, & n=2\\ 0, & otherwise\end{cases}$$

$$y[n]=x[n]+\frac{1}{2}x[n-1]$$

(1) 求impulse response $h[n]=?$ (2) 求$y[n]=?$

$$(1)\text{若}x[n]=\delta[n],\quad y[n]=H\{x[n]\}=x[n]+\frac{1}{2}x[n-1]$$

$$\xrightarrow{x[n]=\delta[n]} h[n]=H\{\delta[n]\}=\delta[n]+\frac{1}{2}\delta[n]$$

(2) $x[n]=2\delta[n]+4\delta[n-1]-2\delta[n-2]$

$$x[n]=2\delta[n]+4\delta[n-1]-2\delta[n-2]$$

$$y[n]=H\{x[n]\}$$

$$=H\{2\delta[n]+4\delta[n-1]-2\delta[n-2]\}$$

$$\xrightarrow{H} y[n]=2h[n]+4h[n-1]-2h[n-2]$$

$$\xrightarrow{LTI} y[n]=\begin{cases}0, & n<0\\ 2, & n=0\\ 5, & n=1\\ 0, & n=2\\ -1, & n=3\\ 0, & n\geq 4\end{cases}$$

Convolution Sum運算步驟

定義 $x[n]*h[n]=\sum_{k=-\infty}^{\infty}x[k]h[n-k]$

(1)令

$w_n[k]=x[k]h[n-k]$

$\because k$為時間軸, $h[k]$

$\xrightarrow{\text{作法}}$ (1) $h[k]$移至 $-n$, 再反射)

（反射$h[k]$, 再移至n）

(2)訊號乘 $x[k]$

(2)求訊號$w_n[k]$之時間和 $y[n]=\sum_{k=-\infty}^{\infty}w_n[k]$

Ex 2.2

$$h[n]=\left(\frac{3}{4}\right)^n u[n], \qquad x[n]=u[n]$$

$$y[n]=x[n]*h[n],$$

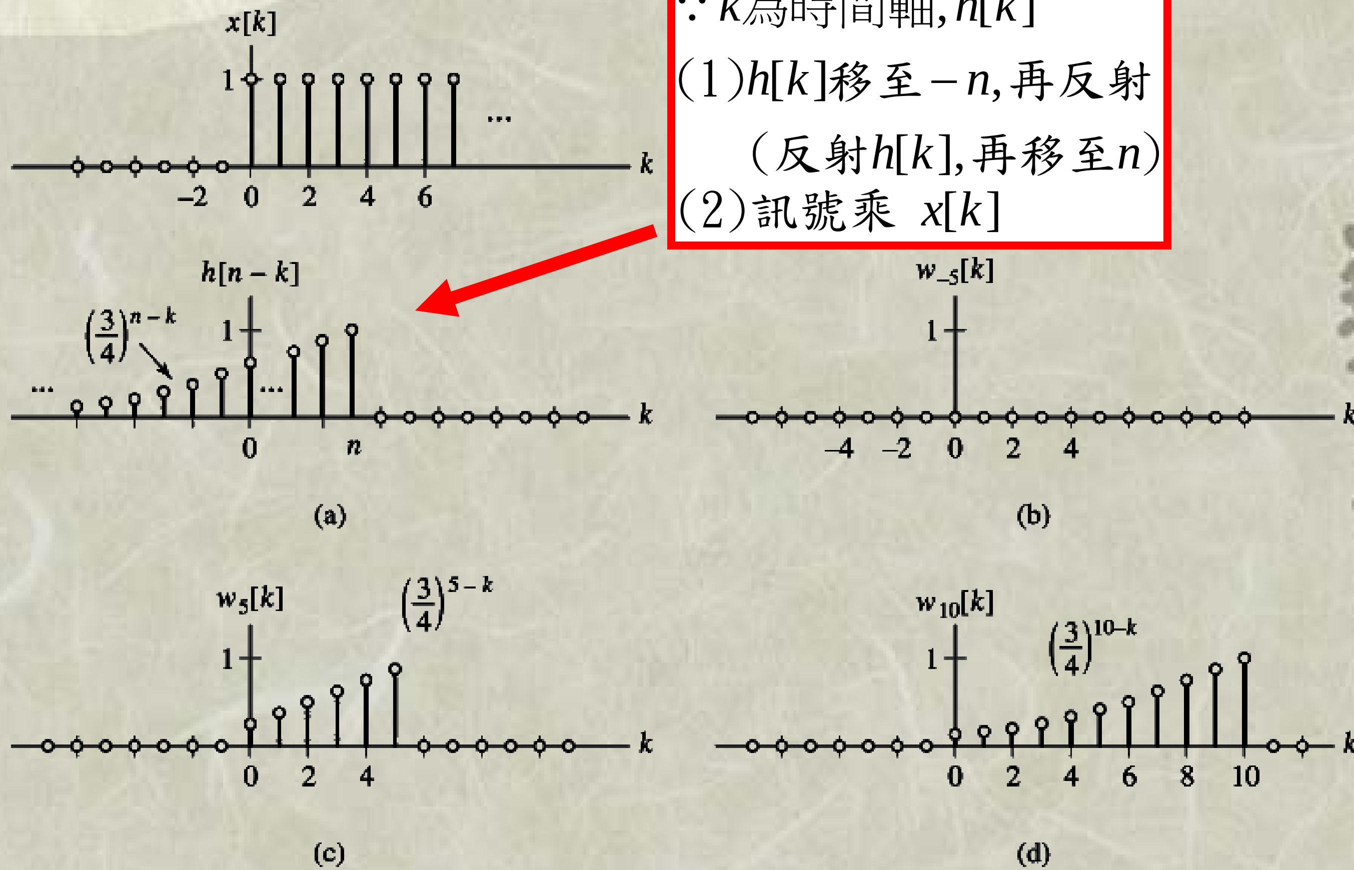
求$y[-5]=?, y[5]=?, y[10]=?$

$\because k$為時間軸, $h[k]$

(1) $h[k]$移至$-n$, 再反射

（反射$h[k]$, 再移至n）

(2) 訊號乘 $x[k]$



Ex 2.2

$(a)\,y[-5]=0,(\because$ 因為$h[n\text{-}k]$尚未移入$x[k]\neq 0$之時間)

$$(b)\,y[5]=\sum_{k=0}^{5}\left(\frac{4}{3}\right)^{5-k}=\left(\frac{3}{4}\right)^{5}\sum_{k=0}^{5}\left(\frac{4}{3}\right)^{k}$$

$$=3.288$$

$$(b)\,y[5]=\sum_{k=0}^{10}\left(\frac{4}{3}\right)^{10-k}=\left(\frac{3}{4}\right)^{10}\sum_{k=0}^{10}\left(\frac{4}{3}\right)^{k}$$

$$=3.831$$

Ex 2.3

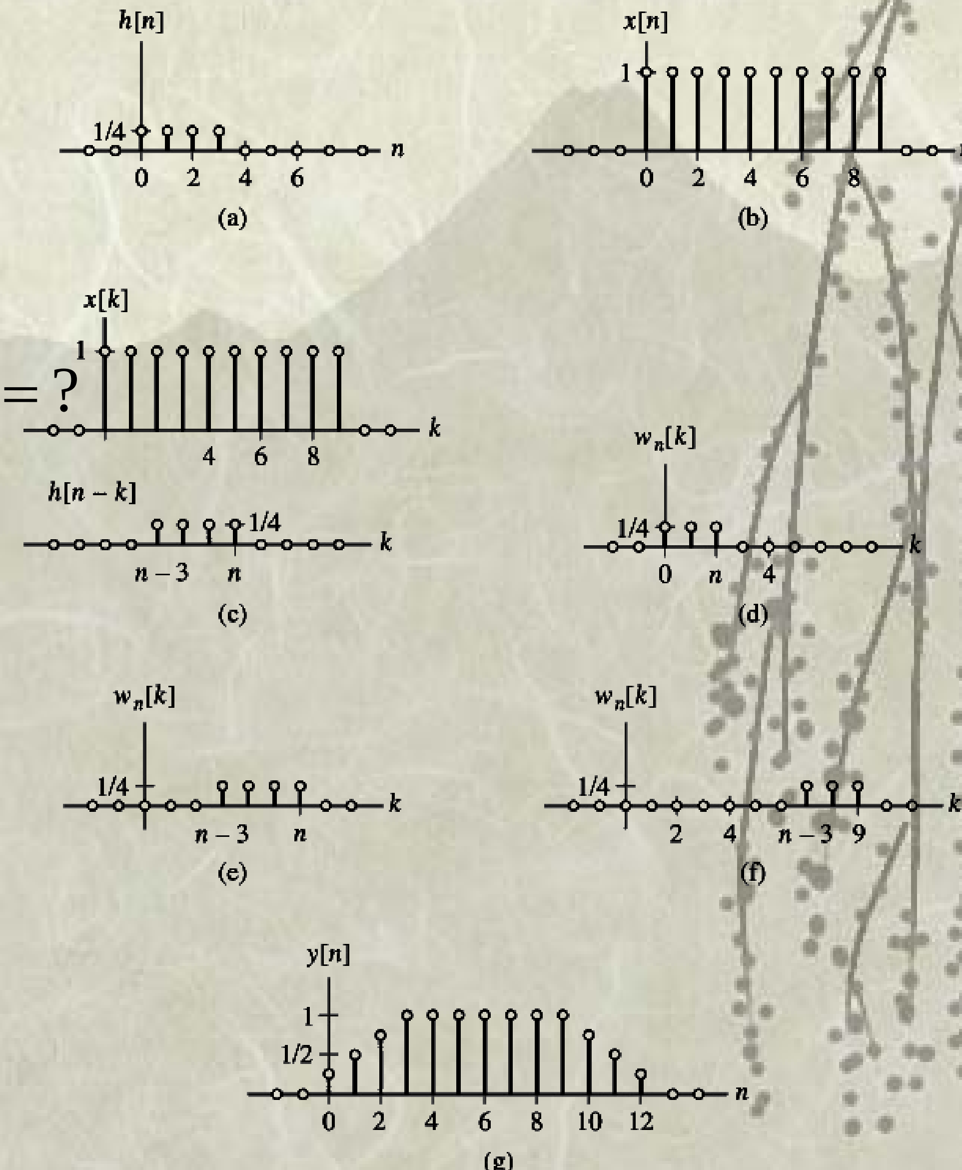
$$y[n]=\frac{1}{4}\sum_{k=0}^{3}x[n-k]$$

$$(1)x[n]=\delta[n],\ y[n]=?$$

$$(2)x[n]=u[n]-u[n-10],\ y[n]=?$$

$$(1)h[n]=H\{\delta[n]\}$$

$$=\frac{1}{4}\sum_{k=0}^{3}\delta[n-k]$$



Ex 2.3

$$(2)h[n-k]=H\{\delta[n-k]\}, \qquad R\{\bullet\}\text{為}reflect$$

$$h[n-k]=h[(-k)-(-n)]=S^n\{h[-k]\}=S^n\{R\{h[k]\}\}$$

$$=S^n\cdot R\cdot H\{\delta(k)\}=S^n\cdot R\left\{\frac{1}{4}\sum_{k'=0}^{3}\delta[k-k']\right\}$$

$$=S^n\left\{\frac{1}{4}\sum_{k'=0}^{3}\delta[-k-k']\right\}=S^n\left\{\frac{1}{4}\sum_{k'=-3}^{0}\delta[-k+k']\right\}$$

$$=\frac{1}{4}\sum_{k'=-3}^{0}\delta[-(k-n)+k']=\frac{1}{4}\sum_{k'=n-3}^{n}\delta[-k+k']$$

Ex 2.3

$$h[n-k]=\frac{1}{4}\sum_{k'=n}^{n+3}\delta[-k+k']$$

$$x[n]=u[n]-u[n-10]$$

$$y[n]=\sum_{k=-\infty}^{\infty}x[k]h[n-k]$$

$$=\sum_{k=-\infty}^{\infty}\left(u[k]-u[k-10]\right)\left(\frac{1}{4}\sum_{k'=n-3}^{n}\delta[-k+k']\right)$$

$$=\sum_{k=0}^{10}\left(\frac{1}{4}\sum_{k'=n-3}^{n}\delta[-k+k']\right)$$

$$h[n-k]=\frac{1}{4}\sum_{k'=n}^{n+3}\delta[-k+k']$$

$$x[n]=u[n]-u[n-10]$$

$$y[n]=\sum_{k=0}^{10}\left(\frac{1}{4}\sum_{k'=n-3}^{n}\delta[-k+k']\right)$$

$$n<0, k'<0, -k+k'<0 \rightarrow y[n]=0$$

$$n=0, \rightarrow y[0]=1/4$$

$$0\le n\le 3, y[n]=\sum_{k=0}^{n}1/4=(n+1)/4$$

$$3<n\le 9, y[n]=\sum_{k=n-3}^{n}1/4=1$$

$$9<n\le 12, y[n]=\sum_{k=n-3}^{9}1/4=(13-n)/4$$

$$12<n, y[n]=0$$

Ex 2.4

$$y[n]-\rho y[n-1]=x[n]$$

$input\quad x[n]=b^n u[n+4], b\neq n.$ System is causal.

求$y[n]=?$

$$(1)\qquad h[n]=H\{\delta[n]\}$$

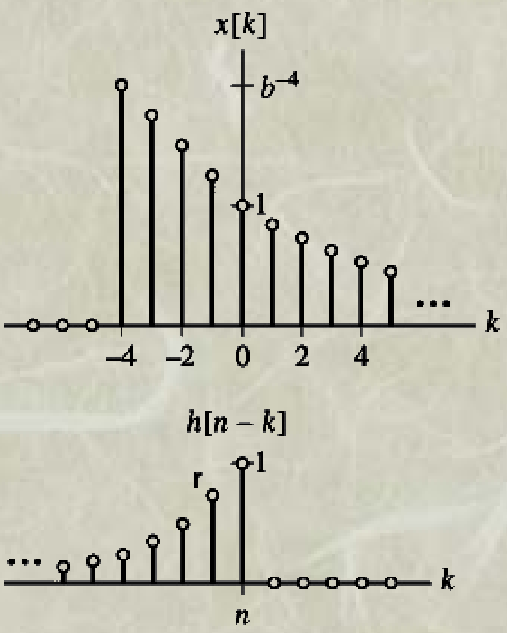
$$\rightarrow h[n]=\rho h[n-1]+\delta[n]$$

$$h[n-1]=\rho h[n-2]+\delta[n-1]$$

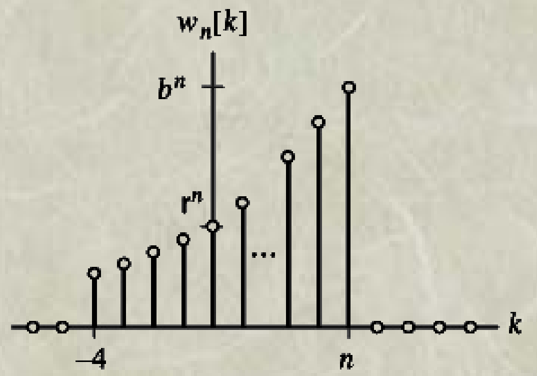
$$\vdots$$

$$h[n-\infty]=\rho h[n-\infty-1]+\delta[n-\infty]$$

$$\rightarrow h[n]=\sum_{k=0}^{\infty}\rho^k\delta[n-k]=\rho^n u[n]$$



(a)



(b)

Ex 2.4

(2) $h[n] = \rho^n u[n]$

$$h[n-k] = \rho^{n-k} u[n-k]$$

$$w_n[k] = x[k]h[n-k]$$

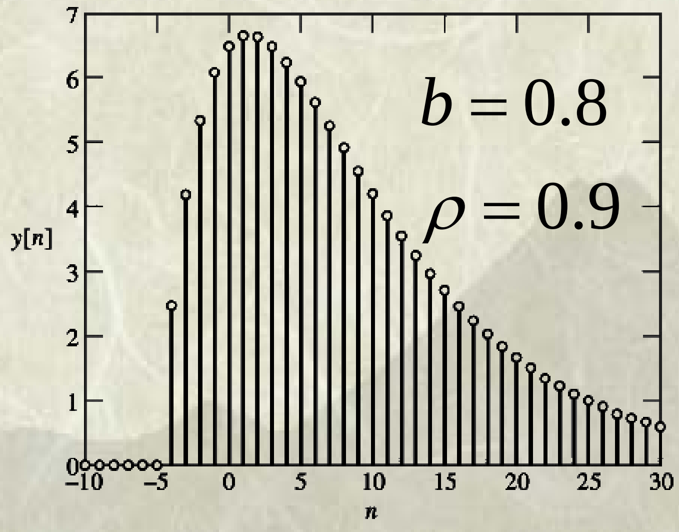
$$= b^k u[k+4]\rho^{n-k} u[n-k],$$

$$= \begin{cases} b^k \rho^{n-k}, & -4 \le k \le n \\ 0, & otherwise \end{cases}$$

$u[k+4] \rightarrow k \ge -4$

$u[n-k] \rightarrow n \ge k$

Ex 2.4



(c)

$$y[n]=\sum_{k=-\infty}^{\infty} w_n[k]=\rho^n\sum_{k=-4}^{n}\left(b^k\rho^{-k}\right)=\rho^n\left(\frac{\rho}{b}\right)^4\sum_{k=0}^{n+4}\left(\frac{b}{\rho}\right)^k$$

$$=\rho^n\left(\frac{\rho}{b}\right)^4\left(\frac{1-\left(\frac{b}{\rho}\right)^{n+5}}{1-\frac{b}{\rho}}\right)$$

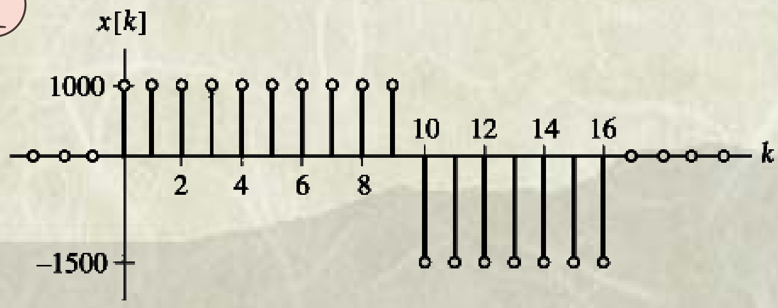
$$=\begin{cases} 0, & n<-4 \\ b^{-4}\left(\dfrac{\rho^{n+5}-b^{n+5}}{\rho-b}\right), & -4\le n \end{cases}$$

Ex 2.5

$$y[n] - \rho y[n-1] = x[n]$$

$$\rho = 1 + \frac{r}{100}, \qquad (r = 8)$$

①



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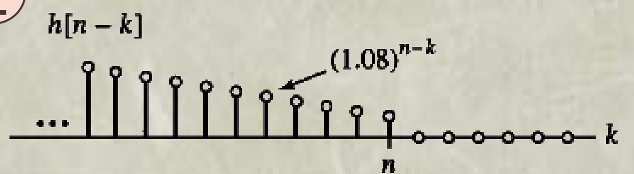
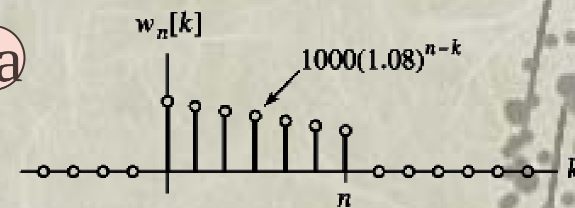


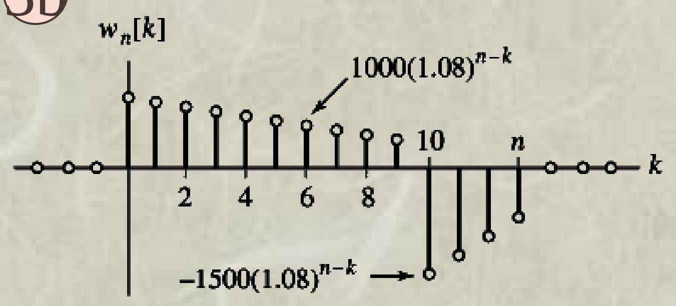
(a)

3a



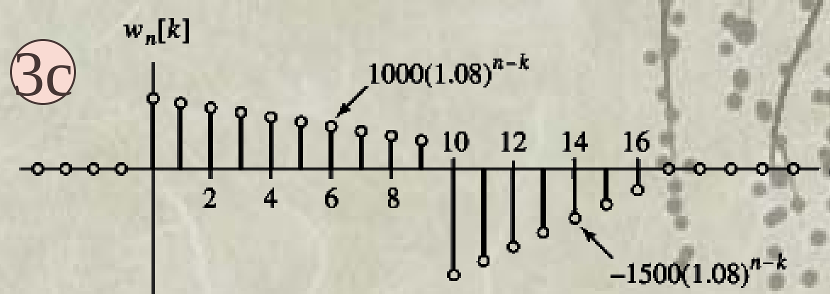
(b)

3b



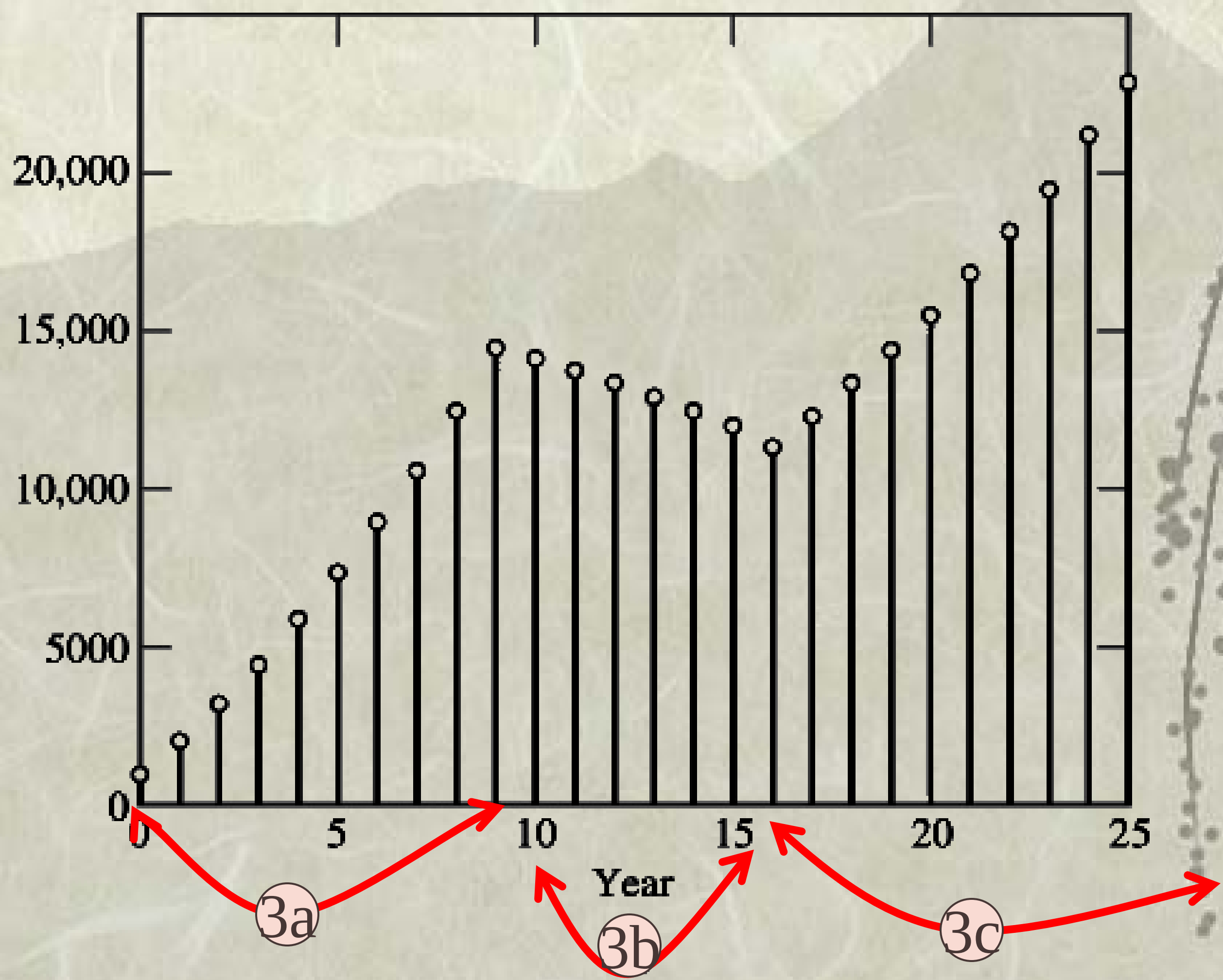
(c)

3c



(d)

Ex 2.5



*Signals for Problem 2.2(f), y[n]=x[n]*h[n]*